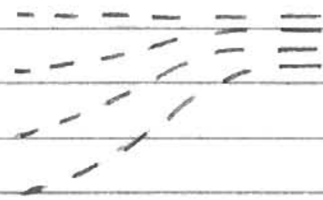
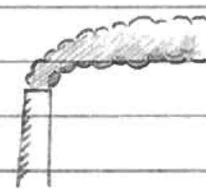
13.42 HOMEWORK #1 SOLUTIONS

1. a. AN IDEAL FLUID IS INCOMPRESSIBLE AND HAS ZERO VISCOSITY.

   b. STREAMLINE: SCATTER BUOYANT PARTICLES ONTO A FLUID SURFACE AND TAKE PHOTOGRAPH WITH SHORT EXPOSURE TIME:



   PATHLINE: LINES IN THE SKY DUE TO METEORITES:

   STREAKLINE:  SMOKE FROM A CHIMNEY

2. a. DISPERSION RELATION IN FINITE DEPTH:

$$\omega^2 = gk \tanh kH$$

   b. THIS TENDS TOWARD $\omega^2 = gk$ FOR $kH \gg 1$.

   c. THE DISPERSION RELATION COMES FROM SUBSTITUTING THE SOL'N $\Phi(x, z, t)$ INTO THE FREE SURFACE COND'N $\Phi_{tt} + g\Phi_z = 0$ on $z = 0$. THE RESULT IS A COND'N THAT RELATES ω & k – i.e., $\omega = f(k)$.

THE DISPERSIVE NATURE OF WATER WAVES MAY BE OBSERVED ON THE OCEAN, WHERE LONG WAVES TRAVEL FASTER THAN SHORTER WAVES.

3. GIVEN $\eta(x,t) = A \sin(kx - \omega t)$,

$$\phi(x,z,t) = -\frac{gA}{\omega} e^{kz} \cos(kx - \omega t) \quad \text{(IN DEEP WATER)}$$

THEN • $u = \frac{\partial \phi}{\partial x} = \omega A e^{kz} \sin(kx - \omega t)$

• $w = \frac{\partial \phi}{\partial z} = -\omega A e^{kz} \cos(kx - \omega t)$

• $p_d = -\rho \frac{\partial \phi}{\partial t} = \rho g A e^{kz} \sin(kx - \omega t)$

a. PLEASE SEE p. 7

b. PLEASE SEE p. 8

c. $p_{tot} = -\rho \frac{\partial \phi}{\partial t} - \rho g z \quad ; \quad z_n = -n \frac{\lambda}{36}$

• AT WAVE CREST, $p_{tot} = \rho g A e^{kz_n} - \rho g z_n$

• AT WAVE TROUGH, $p_{tot} = -\rho g A e^{kz_n} - \rho g z_n$

• AT NODE, $p_{tot} = -\rho g z_n$

d. ACCELERATION: $\frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V}$

HOWEVER, SINCE $\overbrace{(\vec{V} \cdot \nabla)\vec{V}}^{O(\varepsilon^2)} \ll \frac{\partial \vec{V}}{\partial t}$, SAY

$$\frac{DV}{Dt}(x,z,t) \cong \frac{\partial}{\partial t} \vec{V}(x,z,t)$$

• $\frac{\partial u}{\partial t} = -\omega^2 A e^{kz} \cos(kx - \omega t)$

• $\frac{\partial w}{\partial t} = -\omega^2 A e^{kz} \sin(kx - \omega t)$

AS SHOWN ON p. 9, u_t IS 90° OUT-OF-PHASE WITH η AND w_t IS 180° OUT-OF-PHASE WITH η.

e. THE HORIZONTAL VISCOUS FORCE IS PROPORTIONA TO $u|u|$. AS SHOWN ON p. 10, THE MAXIMUM VALUES OF $u|u|$ OCCUR UNDER THE CREST/TROUGH

THE VERTICAL VISCOUS FORCE IS PROPORTIONAL TO $w|w|$. THE MAX VALUES OF $w|w|$ OCCUR AT THE WAVE NODAL POINTS.

4. a. $H = 18$ m

$z_{dome} = -9$ m

$d = 0.6$ m

$\omega = 1$ rad/s $\quad \omega^2 = gk \tanh kH$ (FINITE DEPTH)

FROM J.N. NEWMAN'S MARINE H'DYNAMICS (1977), FIGURE 6.3 (p. 245),

$$\text{FOR } \frac{\omega^2 h}{g} = \frac{1 \cdot 18}{9.81} = 1.83, \quad \frac{h}{\lambda} = \frac{18}{\lambda} \cong 0.28$$

$$\longrightarrow \lambda = 64.29 \text{ m}.$$

$$k = \frac{2\pi}{\lambda} = 0.098 \text{ m}^{-1}$$

FROM EQ. (3.2) IN 13.42 READING # 2, WE KNOW THAT

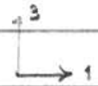
$$p_o = \rho g A \frac{\cosh[k(z+H)]}{\cosh kH}$$

$$19{,}000\ \mathrm{N/m^2} = A\,(1000\ \mathrm{kg/m^3})(10\ \mathrm{m/s^2})(0.4711)$$

b. SEE p. 6.

$$\underline{A = 4.033\ \mathrm{m}}$$

c. THE VERTICAL INERTIAL FORCE DUE TO THE LOCAL ACCELERATION OF THE FLUID RELATIVE TO THE FIXED SPHERE IS



$$F_{I_3} \cong (m_{33} + \rho \forall)\,\frac{\partial w}{\partial t}\Big|_{\substack{x=0\\ z=-9\,\mathrm{m}}}$$

- FROM EQ. (2.10) IN 13.42 READING # 2,

$$\frac{\partial w}{\partial t}(x,z,t) = \omega^2 A \frac{\sinh k(z+H)}{\sinh kH} \cos(kx - \omega t)$$

- FROM J.N. NEWMAN'S M.H., $m_{33} = \frac{2}{3}\rho\pi\left(\frac{d}{2}\right)^3$

- $\forall = \frac{4}{3}\rho\pi\left(\frac{d}{2}\right)^3$

5. a. EULER: $\frac{\partial \vec{V}}{\partial t} + \vec{V}\cdot\nabla\vec{V} = -\frac{1}{\rho}\nabla(p + \rho g z)$

INVOKING THE VECTOR IDENTITY (EQ. (4.3) IN READING 1):

$$\frac{1}{2}\nabla(\vec{V}\cdot\vec{V}) = \vec{V}\cdot\nabla\vec{V} + \vec{V}\times(\nabla\times\vec{V})$$

ASSUME IRROTATIONAL!

$$\frac{\partial V}{\partial t} + \frac{1}{2}\nabla(\vec{V}\cdot\vec{V}) = -\frac{1}{\rho}\nabla(p+\rho g z)$$

[FOR $\nabla \times \vec{V} = 0$, $\vec{V} = \nabla\phi$

$$\frac{\partial}{\partial t}\nabla\phi + \frac{1}{2}\nabla(\nabla\phi\cdot\nabla\phi) = -\frac{1}{\rho}\nabla(p+\rho g z)$$

$$\nabla\left(\frac{\partial\phi}{\partial t}\right) + \nabla\left(\frac{1}{2}\nabla\phi\cdot\nabla\phi\right) = \nabla\left(-\frac{p}{\rho} - gz\right)$$

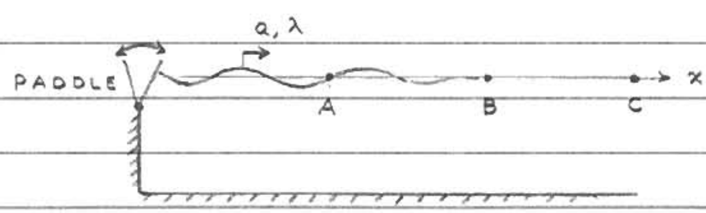
$$\nabla\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \frac{p}{\rho} + gz\right) = 0$$

- FOR $\nabla(\ )$ TO EQUAL ZERO FOR ANY ARBITRARY VALUE OF ( ), ( ) MUST BE INDEPENDENT OF SPACE COORDINATES. THUS,

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \frac{p}{\rho} + gz = C(t).$$

b. $|\nabla\phi|^2 \sim O(\epsilon^2)$, $\therefore$ $p_d = -\rho\frac{\partial\phi}{\partial t}$

6.



a. WAVE FRONT ADVANCES AT $V_g = \frac{1}{2}\frac{\omega}{k}$.

- $k = \frac{2\pi}{\lambda} = 1.5708\ m^{-1}$

- DEEP WATER DISP'N REL.: $\omega^2 = gk \rightarrow \omega = 3.9255\ ^{rad}/_{s}$.

$V_g = \frac{1}{2}\frac{\omega}{k} = 1.25\ ^{m}/_{s}$

THEN $t_A = \frac{x_A}{V_g} = 4\ s$

$t_B = \frac{x_B}{V_g} = 8\ s$

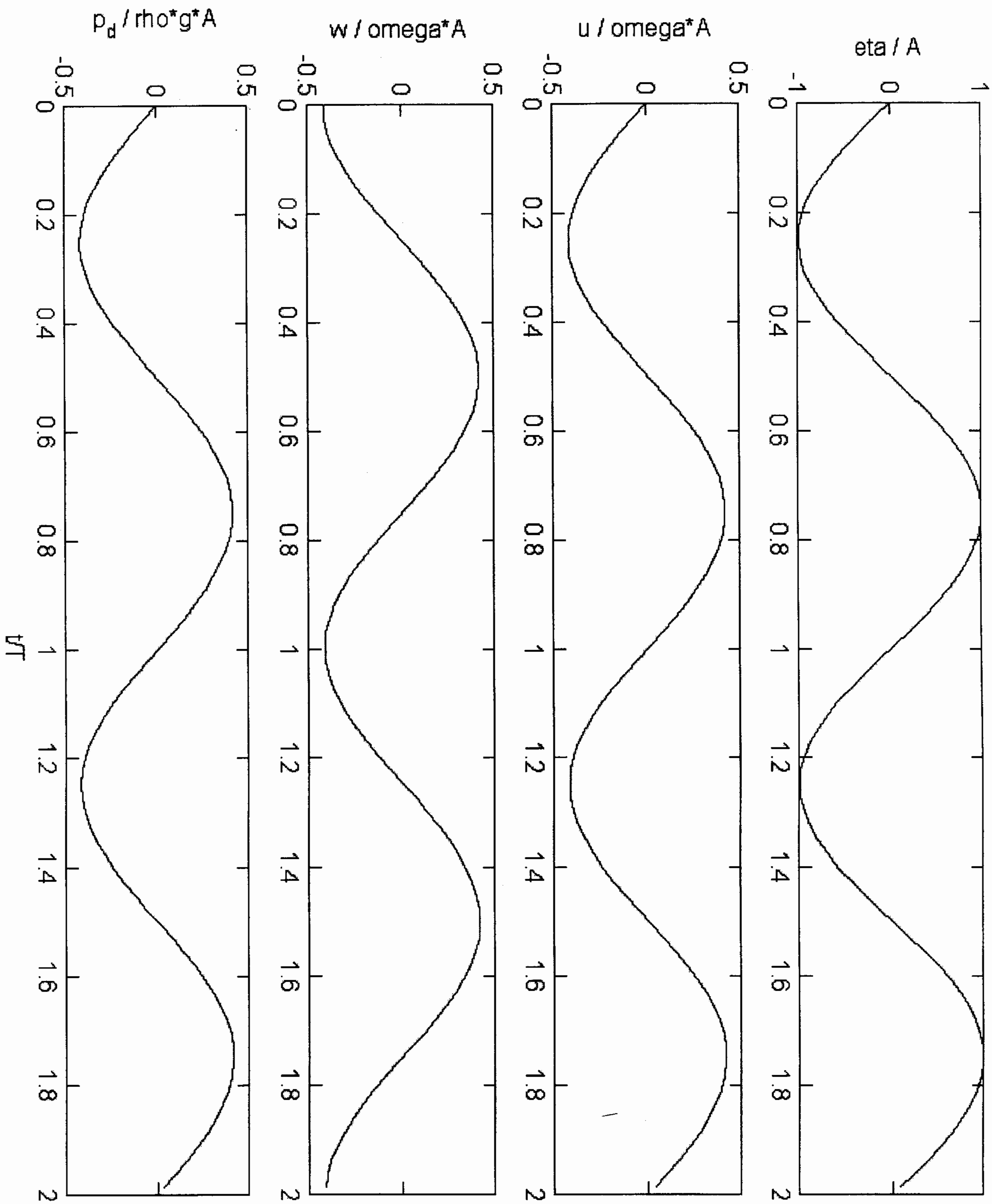
$t_C = \frac{x_C}{V_g} = 12\ s$.

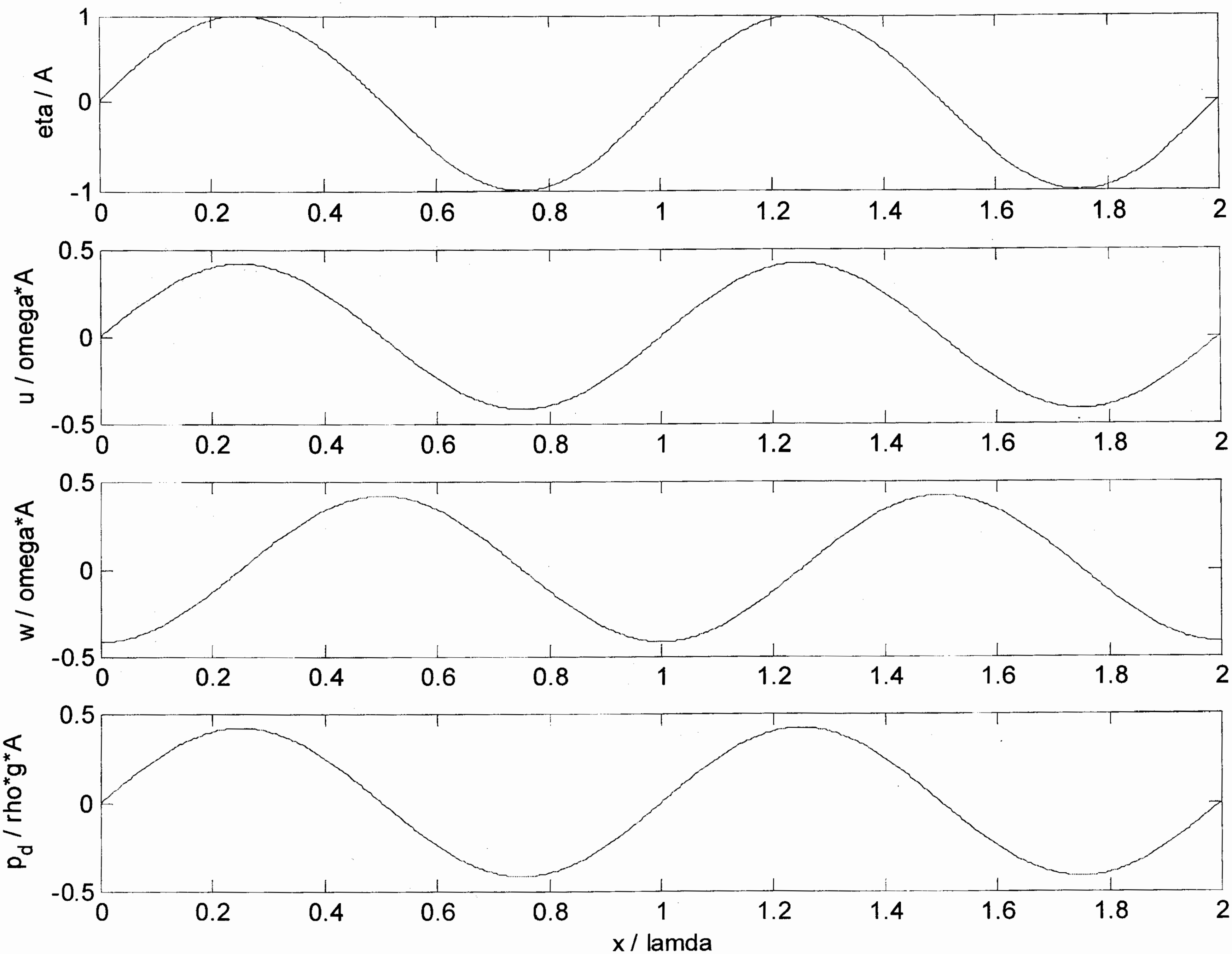
b. YES. IF THEY KNOW THE DISTANCE x FROM THE WAVEMAKER AND TIME t FOR THE WAVE FRONT TO REACH THEM, THEN $V_g = x/t$.
IN DEEP WATER, $V_p = 2\ V_g$.

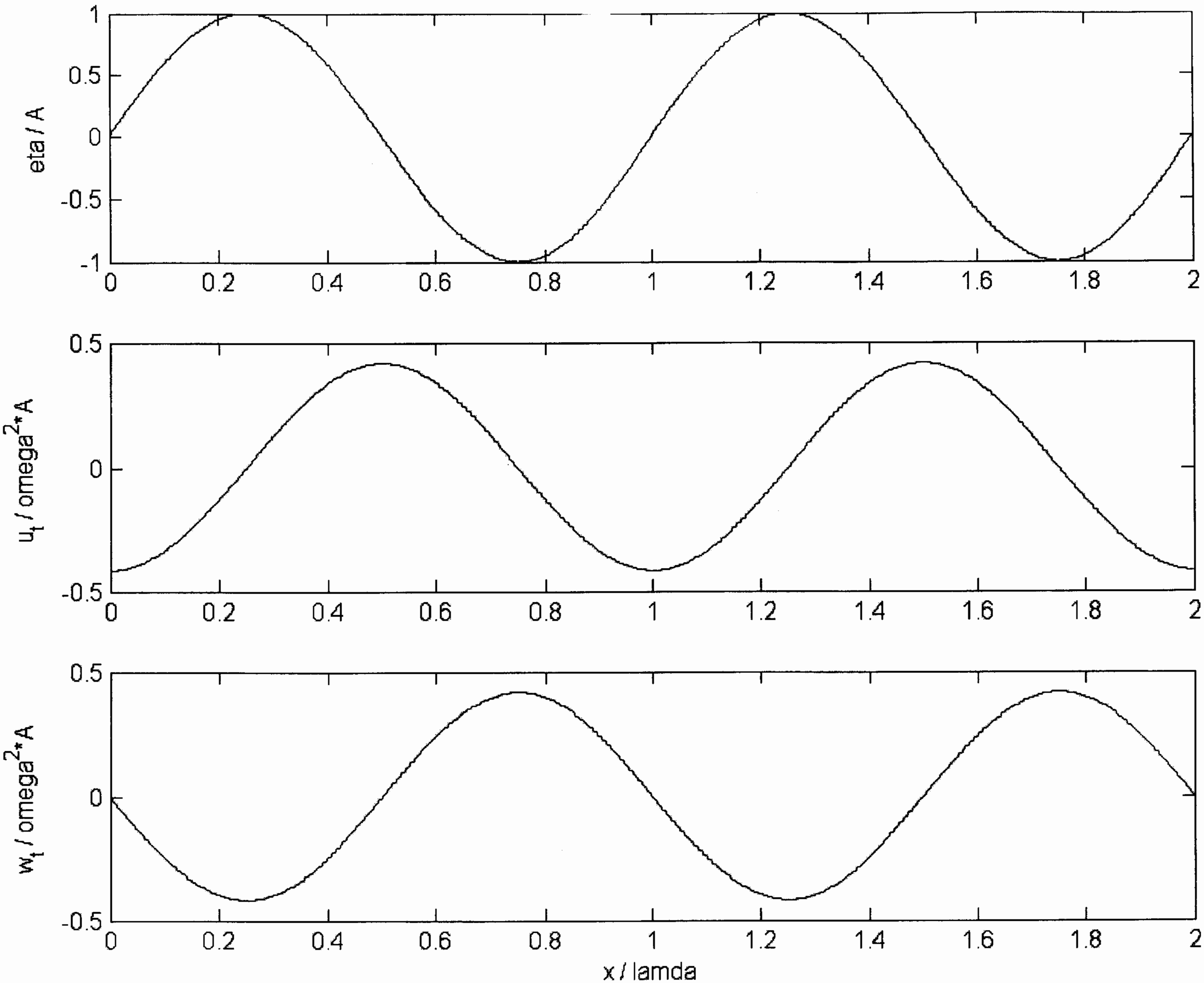
c. THE POS'N DOES NOT MATTER, AS LONG AS x AND t ARE KNOWN.

4b.

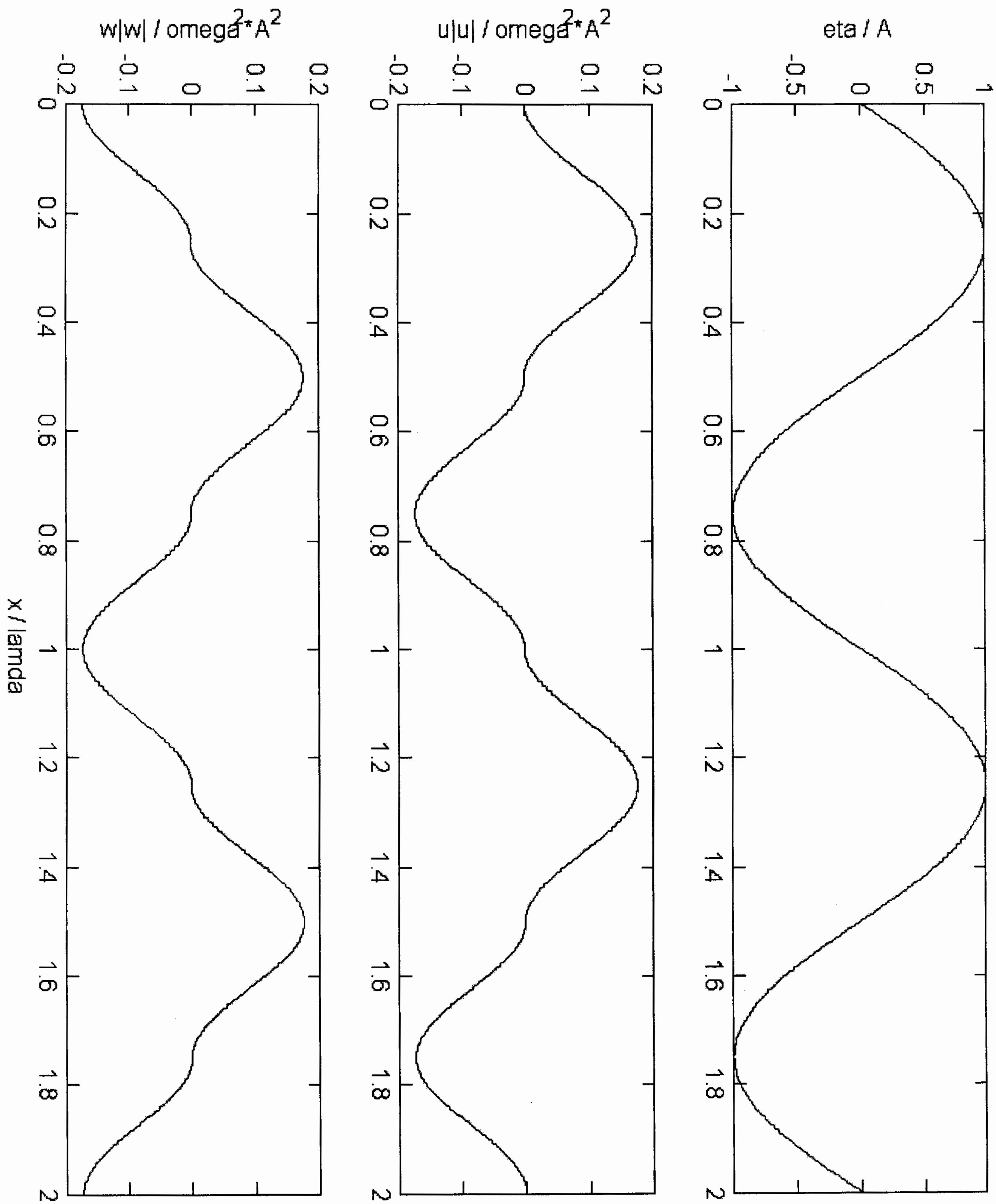
- $\bar{E}_p = \frac{1}{4} \rho g A^2 = 4.0663 \times 10^4\ J/m^2$
- $\bar{E}_k = \frac{1}{4} \rho g A^2 = 4.0663 \times 10^4\ J/m^2$
- $\bar{E} = \frac{1}{2} \rho g A^2 = 8.1325 \times 10^4\ J/m^2$


eta / A
u / omega*A
w / omega*A
p_d / rho*g*A
t/T


eta / A
u / omega*A
w / omega*A
p_d / rho*g*A
x / lamda
1
0
-1
0.5
-0.5
0.2
0.4
0.6
0.8
1.2
1.4
1.6
1.8
2



eta / A
1
0.5
0
-0.5
-1
0 0.2 0.4 0.6 0.8 1 1.2 1.4 1.6 1.8 2
u_t / omega^2*A
0.5
0
-0.5
0 0.2 0.4 0.6 0.8 1 1.2 1.4 1.6 1.8 2
w_t / omega^2*A
0.5
0
-0.5
0 0.2 0.4 0.6 0.8 1 1.2 1.4 1.6 1.8 2
x / lamda


eta / A
u|u| / omega^2*A^2
w|w| / omega^2*A^2
x / lamda