2.20 - Marine Hydrodynamics
Lecture 18

4.9 Turbulent Flow – Reynolds Stress

Assume a flow $\vec{v}$ with a time scale T. *Let* τ denote a time scale $\tau << T$. We can then write for each component of the velocity

$$u_i = \bar{u}_i + u_i' \tag{1}$$

where by definition

$$\bar{u}_i = \frac{1}{\tau}\int_0^\tau u_i dt$$

It immediately follows that

$$\bar{u_i'} = \overline{u_i - \bar{u}_i} = \bar{u}_i - \bar{u}_i = 0, \text{ also } \frac{\partial}{\partial x}\bar{u}_i = \overline{\frac{\partial u_i}{\partial x}} \text{ etc.}$$

Substitute Eq. (1) into continuity and average over τ, i.e., take $\overline{(\quad)}$

$$\overline{\frac{\partial u_i}{\partial x_i}} = \overline{\frac{\partial \bar{u}_i}{\partial x_i}} + \underbrace{\overline{\frac{\partial u_i'}{\partial x_i}}}_{\downarrow \, 0} = 0, \qquad \Longrightarrow \boxed{\frac{\partial \bar{u}_i}{\partial x_i} = 0}$$

$$\text{but} \qquad \frac{\partial u_i}{\partial x_i} = 0 = \underbrace{\frac{\partial \bar{u}_i}{\partial x_i}}_{\downarrow \, 0 \text{ , just shown}} + \frac{\partial u_i'}{\partial x_i}, \qquad \Longrightarrow \boxed{\frac{\partial u_i'}{\partial x_i} = 0}$$

Substitute Eq. (1) into the momentum equations and take $\overline{(\ \ )}$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i$$

$$\overline{\frac{\partial u_i}{\partial t}} = \frac{\partial \bar{u}_i}{\partial t} + \underbrace{\overline{\frac{\partial u_i'}{\partial t}}}_{0}; \text{ similarly } \begin{cases} \overline{\nu \nabla^2 u_i} = \nu \nabla^2 \overline{u_i} \\ \\ \overline{\frac{\partial p}{\partial x_i}} = \frac{\partial}{\partial x_i}\overline{(\bar{p} + p')} = \frac{\partial \bar{p}}{\partial x_i} \text{ etc.} \end{cases}$$

$$\overline{u_j \frac{\partial u_i}{\partial x_j}} = \overline{\left(\bar{u}_j + u_j'\right)\frac{\partial}{\partial x_j}\left(\bar{u}_i + u_i'\right)} = \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} + \underbrace{\overline{u_j' \frac{\partial \bar{u}_i}{\partial x_j}}}_{0} + \underbrace{\overline{\bar{u}_j \frac{\partial u_i'}{\partial x_j}}}_{0} + \overline{u_j' \frac{\partial}{\partial x_j} u_i'}$$

but from continuity we have

$$\overline{u_j' \frac{\partial}{\partial x_j} u_i'} = \frac{\partial}{\partial x_j}\overline{u_j' u_i'} - \overline{u_i' \underbrace{\frac{\partial u_j'}{\partial x_j}}_{0 \rightarrow \text{by continuity}}}$$

and thus we finally obtain

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \nabla^2 \bar{u}_i}_{\frac{1}{\rho}\frac{\partial}{\partial x_j}\overline{\tau_{ij}}} - \frac{\partial}{\partial x_j}\overline{u_i' u_j'}$$

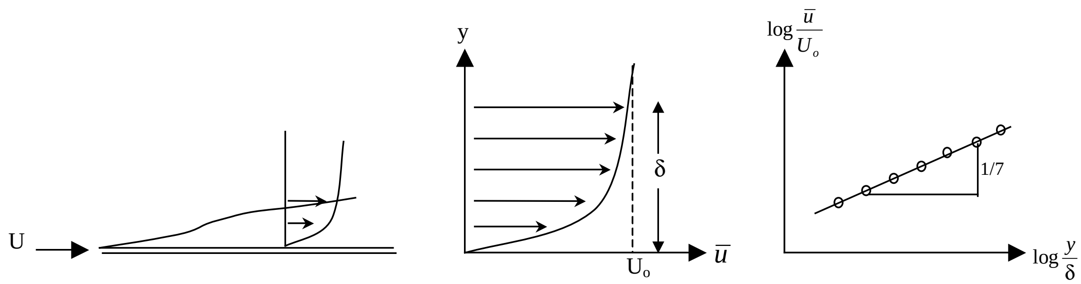
Reynolds averaged N-S equation: $\boxed{\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = \frac{1}{\rho}\frac{\partial}{\partial x_j}\left[\overline{\tau_{ij}} - \rho \overline{u_i' u_j'}\right]}$

Reynolds stress: $\boxed{\tau_{R_{ij}} \equiv -\rho \overline{u_i' u_j'}}$

4.10 Turbulent Boundary Layer Over a Smooth Flat Plate

We have already seen that the function of the friction coefficient $C_f(R_{e_L})$ differs for laminar and turbulent flows. In this paragraph we will discuss the case of a turbulent boundary layer.

Following a procedure similar to that for flow past a body of general geometry, we will use an *approximate* velocity profile, obtain the P-Flow solution and eventually substitute everything into von Karman's momentum integral equation. The velocity profiles used in practice are either empirical ($(1/7)^{th}$ power) or semi-empirical (logarithmic) *laws.*



4.10.1 $(1/7)^{th}$ Power Velocity Profile Law

Let the velocity profile be determined by the following *empirical law*

$$\boxed{\frac{\bar{u}}{U_o} = \left(\frac{y}{\delta}\right)^{1/7}} \tag{2}$$

where $\delta = \delta(x)$ is to be determined.

From equation (2) we can obtain directly $\delta^*$and θ

$$\begin{aligned} \delta^* &= \frac{\delta}{8} \\ \theta &= \frac{7}{72}\delta \cong 0.0972\ \delta \end{aligned}$$

However, we need to use an additional empirical law to determine the skin friction. From Blasius' law of friction for pipes we obtain an expression for τ_o

$$\frac{\tau_o}{\rho U_o^2} = 0.0227 \left(\frac{U_o \delta}{\nu}\right)^{-1/4}$$

From P-Flow for flow past a flat plate we have $U(x) = U_0 = const$, and $dp/dx = 0$
Substituting $\delta^*, \theta, \tau_o, U_o$ into von Karman's moment equation

$$\frac{\tau_o}{\rho U_o^2} = \frac{d}{dx}(\theta) \Longrightarrow 0.0227 \left(\frac{U_o \delta}{\nu}\right)^{-1/4} = \frac{7}{72}\frac{d\delta}{dx}$$

This is a 1^st order ODE for δ. One BC is required. We *assume* that the the flow is tripped at $x = 0$, i.e., at $x = 0$ the flow is already turbulent. Further on, we *assume* that the turbulent boundary layer starts at $x = 0$, i.e., $\delta(0) = 0$. It follows that

$$\delta(x) \cong 0.373x \left(\frac{U_o x}{\nu}\right)^{-1/5} \Longrightarrow \frac{\delta}{x} \cong 0.373 R_{e_x}^{-1/5}$$

Compare:

Laminar Boundary Layer	**Turbulent Boundary Layer** ($1/7^{th}$ power law)
$\delta(x) \propto \sqrt{x}$	$\delta(x) \propto x^{4/5}$
$\delta^* \cong 1.72\sqrt{\frac{\nu x}{U_o}}$	$\delta^* \cong 0.047\left(\frac{\nu x^4}{U_o}\right)^{1/5}$

Once the profile has been determined we can evaluate the friction drag

$$D = 0.036\left(\rho U_o^2\right) BL\ R_{e_L}^{-1/5}$$

Thus, the friction coefficient for turbulent (tripped and/or $R_{e_L} > 5 \times 10^5$) flow over a flat plate is

$$\boxed{C_f = \frac{\mathrm{D}}{\frac{1}{2}\rho U_o^2 BL} = 0.073 R_{e_L}^{-1/5}}$$

4.10.2 Logarithmic Velocity Profile Law

If the velocity profile is determined by the semi-empirical logarithmic velocity profile law, following an approach similar to that for the $1/7^{th}$ power law, we obtain **Schoenherr's** formula for the friction coefficient

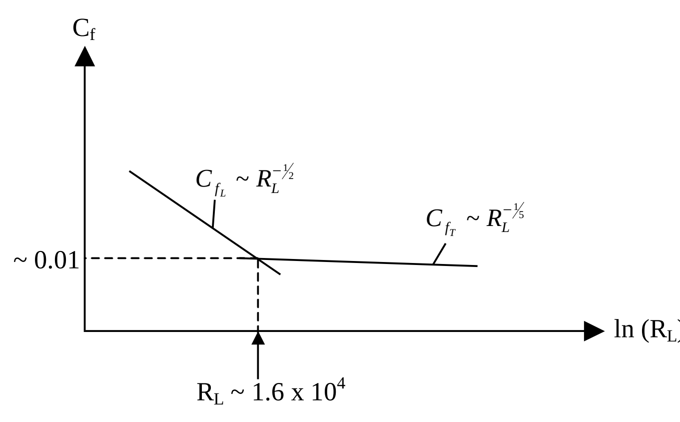
$$\boxed{\frac{0.242}{\sqrt{C_f}} = \log_{10}\left(R_{e_L} C_f\right)}$$

4.10.3 Summary of Boundary Layer Over a Flat Plate

Laminar BL (Blasius)	**Turbulent BL** ($1/7^{th}$ power law)
$\frac{\delta}{x} \propto R_{e_x}^{-1/2}$ $\delta^* = 1.72 x R_{e_x}^{-1/2} \propto \sqrt{x}$	$\frac{\delta}{x} \propto R_{e_x}^{-1/5}$ $\delta^* = 0.047 x R_{e_x}^{-1/5} \propto x^{4/5}$
$\tau_o = 0.332 \rho U_o^2 R_{e_x}^{-1/2}$	$\tau_o = 0.0227 \rho U_o^2 R_{e_\delta}^{-1/4}$ $\tau_o = 0.02297 \rho U_o^2 R_{e_x}^{-1/5}$
$D = 0.664 \rho U_0^2 (BL) R_{e_L}^{-1/2}$	$D = 0.03625 \rho U_0^2 (BL) R_{e_L}^{-1/5}$
$C_f \equiv \frac{D}{\rho U_o^2 (BL)} = 1.328 R_{e_L}^{-1/2}$	$C_f \equiv \frac{D}{\rho U_o^2 (BL)} = 0.0725 R_{e_L}^{-1/5}$

For τ_o, the cross-over is at $R_{e_x} \sim 3.4 \times 10^3$, i.e.,

$$(\tau_o)_{\text{laminar}} > (\tau_o)_{\text{turbulent}} \text{ for } R_{e_x} < 3.4 \times 10^3$$
$$(\tau_o)_{\text{laminar}} \sim (\tau_o)_{\text{turbulent}} \text{ for } R_{e_x} \sim 3.4 \times 10^3$$
$$(\tau_o)_{\text{laminar}} < (\tau_o)_{\text{turbulent}} \text{ for } R_{e_x} > 3.4 \times 10^3$$



Therefore, for most prototype scales:

$$(C_f)_{\text{turbulent}} > (C_f)_{\text{laminar}}$$
$$(\tau_o)_{\text{turbulent}} > (\tau_o)_{\text{laminar}}$$