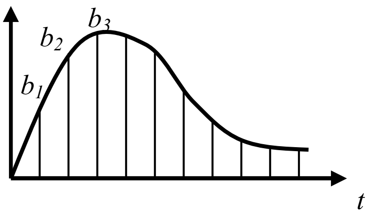
2.160 System Identification, Estimation, and Learning

Lecture Notes No. 11

March 15, 2006

6.5 Times-Series Data Compression

Finite Impulse Response Model



Consider a FIR Model

$$y(t) = b_1 u(t-1) + b_2 u(t-2) + \cdots\cdots + b_m u(t-m)$$

The transfer functions of (11) are then: $G(q) = B(q), \quad H(q) = 1$

One-Step-Ahead Predictor

$$\begin{aligned}\hat{y}(t|\theta) &= H^{-1}(q)G(q)u(t) + \left[1 - H^{-1}(q)\right]y(t) \\ &= G(q)u(t)\end{aligned}$$

$$\hat{y}(t|\theta) = \varphi^T(t)\theta : \text{linear regression}$$

Given input data $\{u(t), 1 \le t \le N\}$, the least square estimate of the parameter vector was obtained as

$$\hat{\theta}_N^{LS} = \arg\min_{\theta} \frac{1}{N}\sum_{t=1}^{N}\frac{1}{2}\left(y(t) - \hat{y}(t \mid \theta)\right)^2 = \left(R(N)\right)^{-1} f(N) \tag{40}$$

where

$$R(N) = \frac{1}{N}\sum_{t=1}^{N}\varphi(t)\varphi(t)^T = \Phi\Phi^T \quad \text{and} \quad f(N) = \frac{1}{N}\sum_{t=1}^{N}\varphi(t)y(t) \tag{41}$$

Pro's and Con's of FIR Modeling

Pros.

LSE gives a consistent estimate $\lim_{N\to\infty}\hat{\theta}_N^{LS} = \theta_0$ as long as the input sequence $\{u(t)\}$ is uncorrelated with the noise *e(t)*, which may be colored.

Cons

Two failure scenarios

- The impulse response has a slow decaying mode

- The sampling rate is high

The number of parameters, *m,* is too large to estimate.
The persistently exciting condition $rank\Phi = full\ rank$ can hardly be satisfied.
Check the eigenvalues of $\Phi\Phi^T$ (or the singular values of Φ)

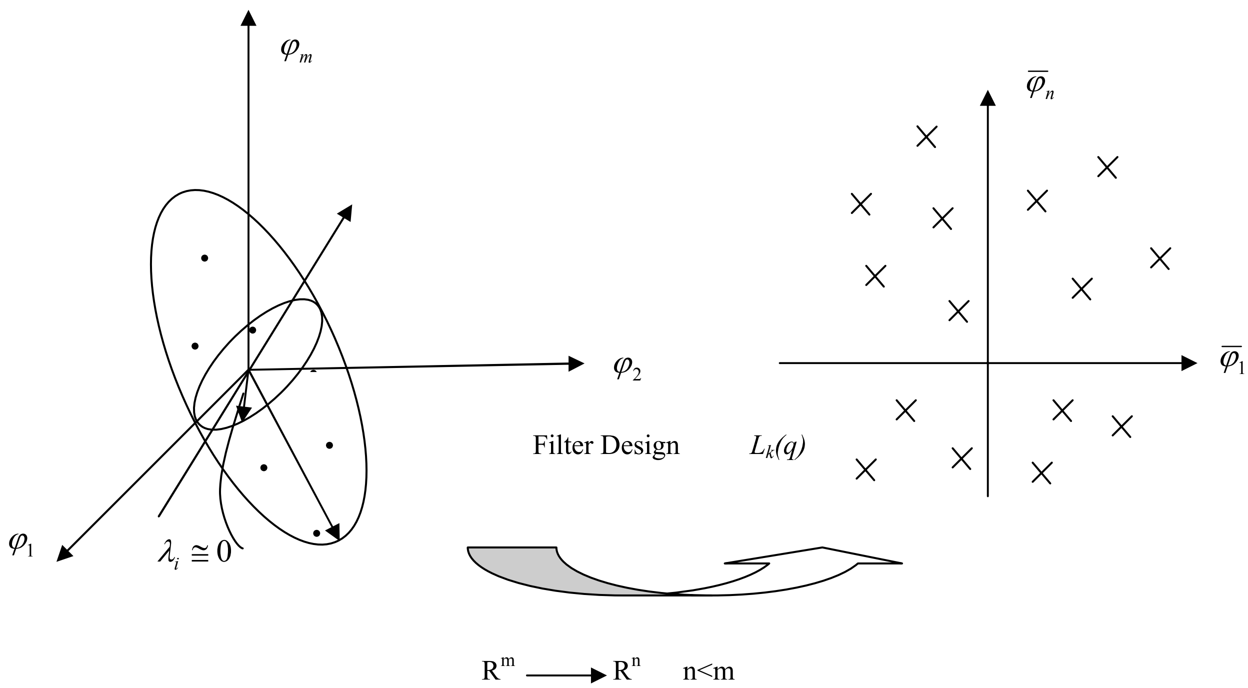
$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$$

It is likely

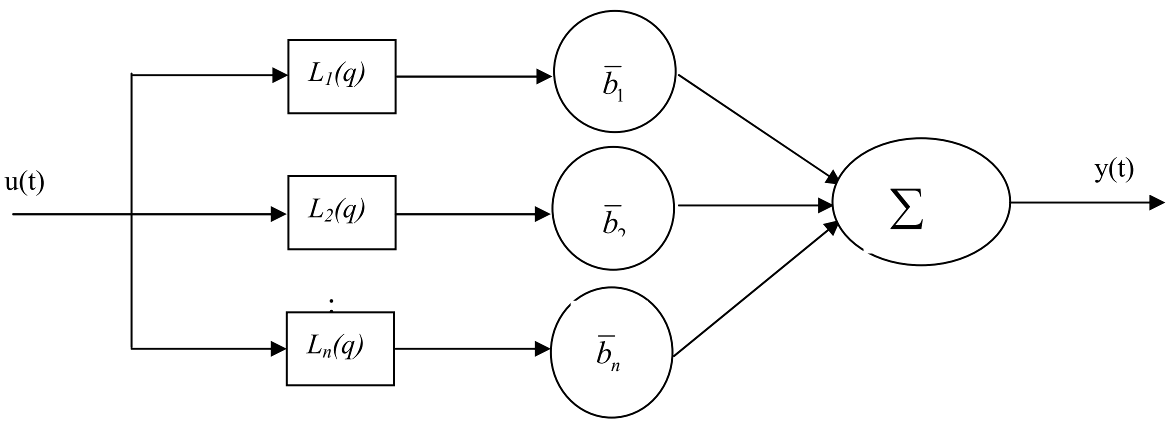
$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \lambda_{n+1} \cong 0 = \cdots = \lambda_m$$

Often m becomes more than 50 and it is difficult to obtain such an input series having 50 non-zero singular values.

Time series data compression is an effective method for coping with this difficulty. Before formulating the above least square estimate problem, data are processed so that the information contained in the series of regressor may be represented in compact, low-dimensional form.



Coordinate Transformation (Data Compression)



6.6 Continuous-Time Laguerre Series Expansion

Let us begin with continuous-time Laguerre expansion.

[Theorem 6.6]
Assume a transfer function $G(s)$ is

- Strictly proper $\lim_{s\to\infty} G(s) = 0 \quad G(\infty) = 0$ (45)
- Analytic in $\mathrm{Re}(s) > 0$ No pole on RHP
- Continuous in $\mathrm{Re}(s) \geq 0$

Then, there exists a sequence $\{\bar{g}_k\}$ such that

$$G(s) = \sum_{k=1}^{\infty} \bar{g}_k \frac{\sqrt{2\bar{a}}}{s+\bar{a}} \left(\frac{s-\bar{a}}{s+\bar{a}} \right)^{k-1} \tag{46}$$

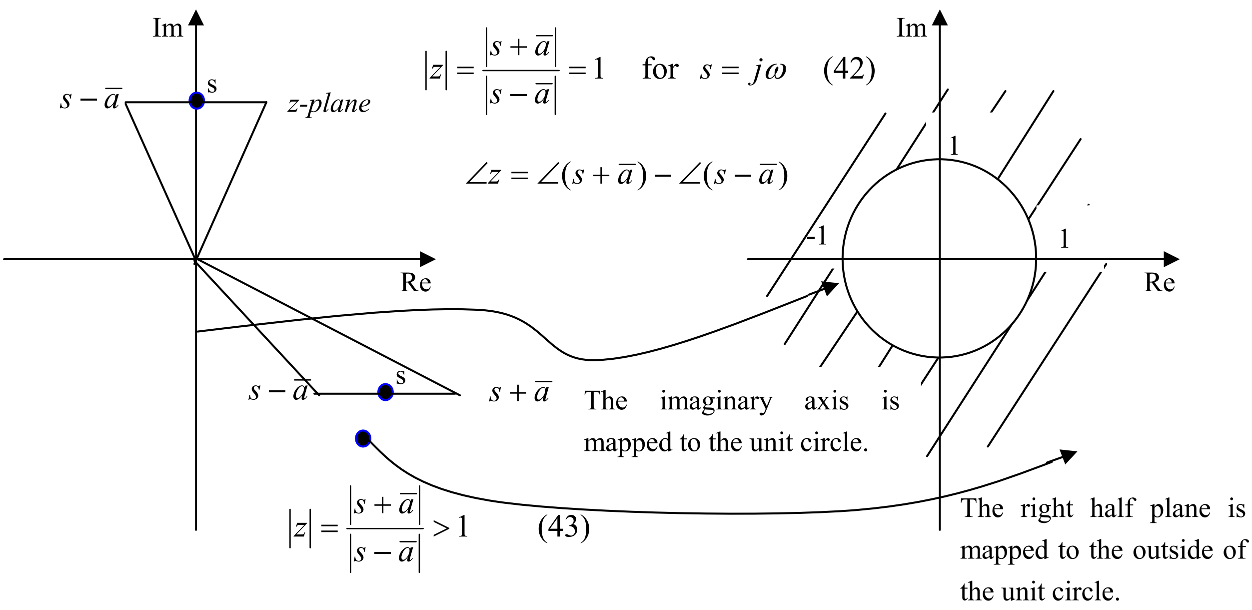
where $\bar{a} > 0$

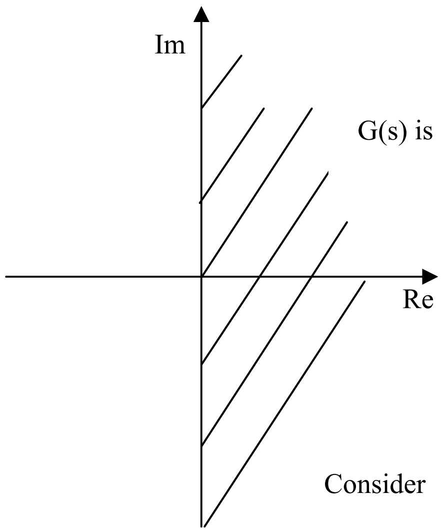
<u>Proof:</u>
Consider the transformation given by

$$z = \frac{s+\bar{a}}{s-\bar{a}} \tag{47}$$

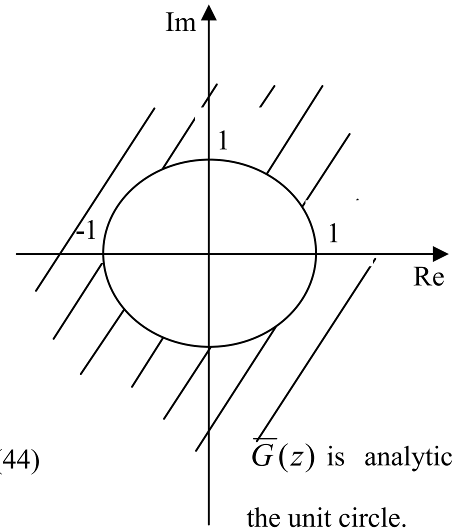
$$sz - \bar{a}z = s + \bar{a} \longrightarrow s(z-1) = \bar{a}(z+1)$$

Bilinear transformation $\therefore s = \bar{a}\dfrac{z+1}{z-1}$ (48)





G(s) is analytic in $\mathrm{Re}(s) > 0$



Consider $\overline{G}(z) = G\left(\underbrace{\overline{a}\,\frac{z+1}{z-1}}_{s}\right)$ (44)

$\overline{G}(z)$ is analytic outside the unit circle.

Therefore, there exists a Laurent expansion for $\overline{G}(z)$:

$$\overline{G}(z) = \sum_{k=1}^{\infty} g_k z^{-k} \qquad |z| > 1 \tag{49}$$

Considering that $G(\infty) = 0 \longrightarrow \lim_{s\to\infty} \frac{s+\overline{a}}{s-\overline{a}} = 1 \quad \therefore \overline{G}(z) = 0$ at z=1

$$\overline{G}(z) = \frac{1}{\sqrt{2\overline{a}}}(1 - z^{-1}) \sum_{k=1}^{\infty} \overline{g}_k z^{-(k-1)} \tag{50}$$

Substituting (40) into (46)

$$\overline{G}\left(\frac{s+\overline{a}}{s-\overline{a}}\right) = G(s) = \frac{1}{\sqrt{2\overline{a}}}(1 - \frac{s-\overline{a}}{s+\overline{a}}) \sum_{k=1}^{\infty} \overline{g}_k \left(\frac{s+\overline{a}}{s-\overline{a}}\right)^{-(k-1)}$$

$$\therefore G(s) = \sum_{k=1}^{\infty} \overline{g}_k \frac{\sqrt{2\overline{a}}}{s+\overline{a}} \left(\frac{s-\overline{a}}{s+\overline{a}}\right)^{(k-1)}$$

$$L_k(s) = \frac{\sqrt{2\overline{a}}}{s+\overline{a}} \left(\frac{s-\overline{a}}{s+\overline{a}}\right)^{(k-1)} \tag{51}$$

Low Pass Filter $\qquad \left|\frac{s-\overline{a}}{s+\overline{a}}\right| \longrightarrow \left|\frac{j\omega-\overline{a}}{j\omega+\overline{a}}\right| = 1$ All-pass Filter

The Laplace transform of the Laguerre functions

The Main Point

The above Laguerre series expansion can be used for data compression if the parameter $\overline{a}$, called a Laguerre pole, is chosen such that slow poles, i.e. dominant poles, of the original system are close to the Laguerre pole. Let p_i be a slow real pole of the original transfer function $G(s)$. If the Laguerre pole $\overline{a}$ is chosen such that

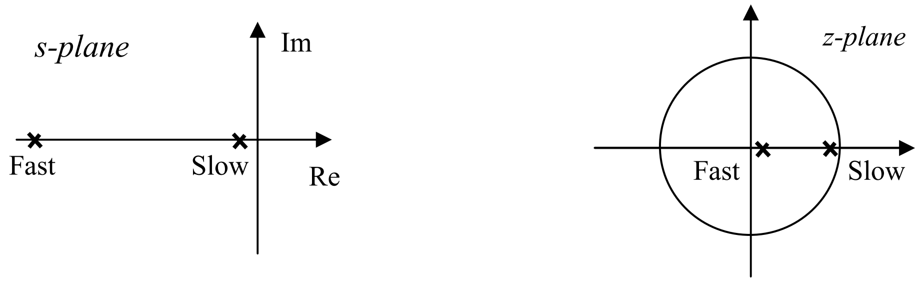
$\bar{a} \cong |p_i|$, then a truncated Laguerre expansion:

$$G_n(s) = \sum_{k=1}^{n} \bar{g}_k \frac{\sqrt{2\bar{a}}}{s+\bar{a}} \left(\frac{s-\bar{a}}{s+\bar{a}} \right)^{k-1} \rightarrow \quad G(s) \tag{52}$$

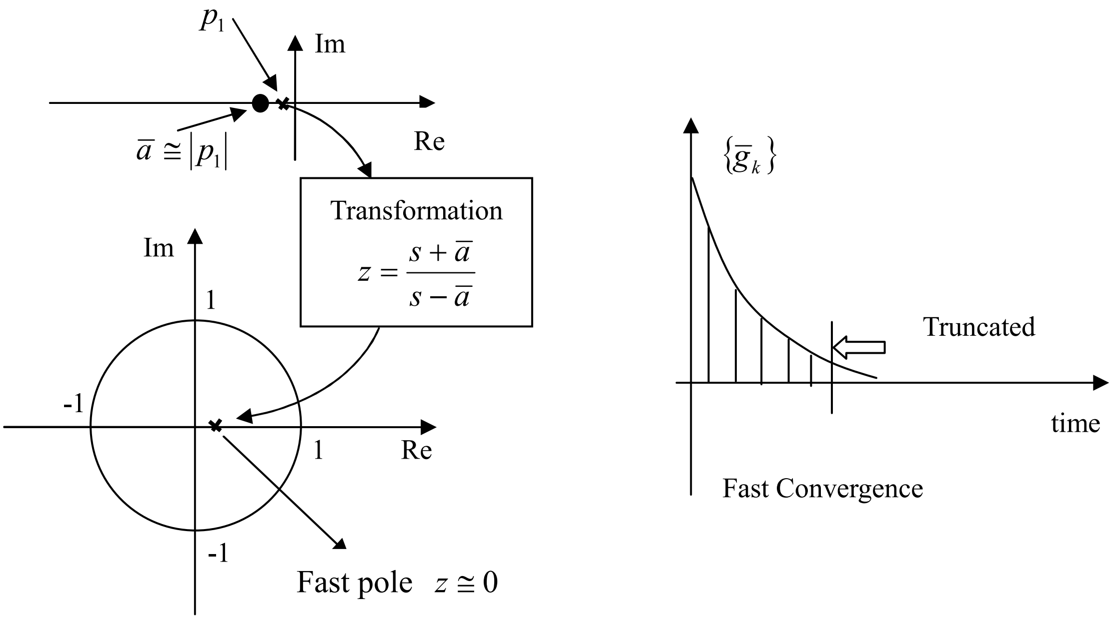
converges to the original G(s) quickly.

- Recall -

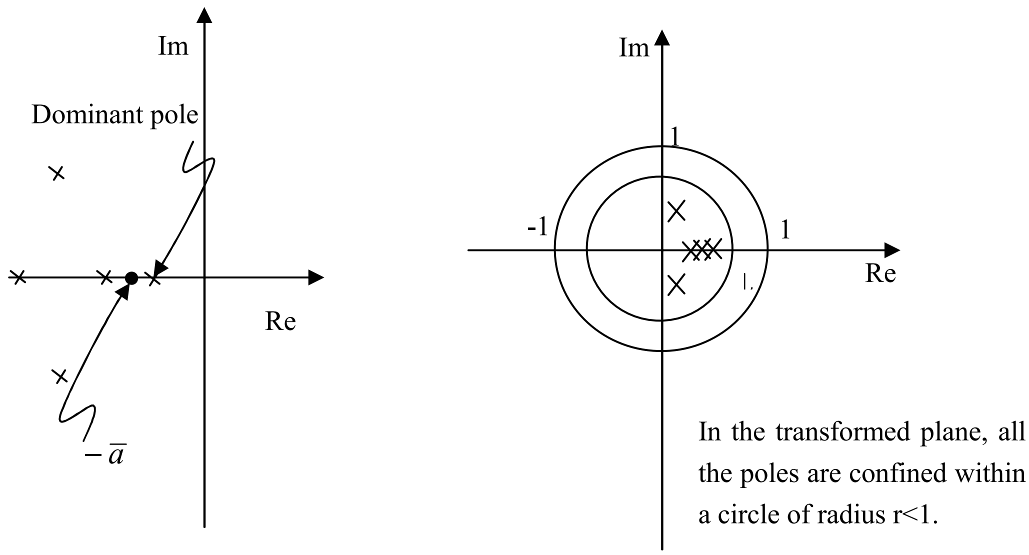
For a continuous-time system, a pole close to the imaginary axis is slow to converge, while a pole far from the imaginary axis converges quickly. Likewise, in discrete time, a pole close to the origin of a z-plane quickly converges, while the ones near the unit circle are slow.



Choose $\bar{a}$ such that $p_1 \cong -\bar{a}$, where p_1 is a slow, stable pole. Using the bilinear transform, this slow pole in the s-plane can be transferred to a fast pole in the z-plane, as shown below. Representing in the z-plane, the transfer function can be truncated; just a few terms can approximate the impulse response since it converges quickly.



When the original transfer function has many poles, the Laguerre pole is placed near the dominant pole so that most of the slow poles may be approximated by the Laguerre pole.



With the bilinear transform, these slow poles are transferred to the ones near the origin of the z-plane. They are fast, hence the transfer function can be approximated to a low order model.

6.7 Discrete-Time Laguerre Series Expansion

[Theorem 6.7]
Assume that a Z-transform G(z) is

- Strictly proper $\quad G(\infty) = 0$
- Analytic in $|z| > 1$ RHP
- Continuous in $|z| \geq 1$

Then

$$G(s) = \sum_{k=1}^{\infty} \bar{g}_k \frac{K}{z-a} \left(\frac{1-az}{z-a} \right)^{k-1} \tag{53}$$

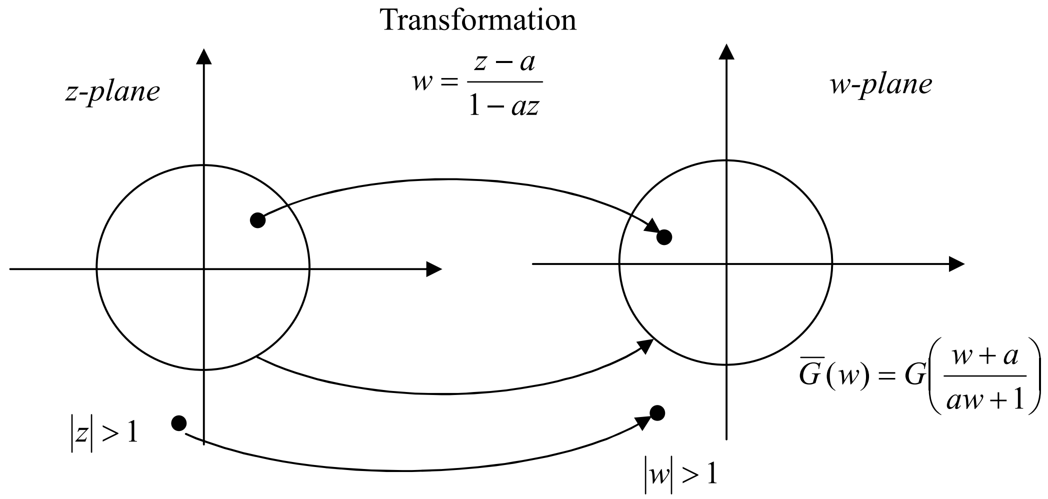
where -1<a<1 and

$$K = \sqrt{(1-a^2)T} \tag{54}$$

<u>Proof</u>
Consider the bilinear transformation:

$$w = \frac{z-a}{1-az} \tag{55}$$

$$w - azw = z - a \qquad w + a = z(aw+1) \text{ therefore, } z = \frac{w+a}{aw+1} \tag{56}$$



$\overline{G}(w) = G\left(\frac{w+a}{aw+1}\right)$ is analytic in $|w| > 1$, and is proper $G(\infty) = 0$

$$\lim_{z\to\infty} w = -\frac{1}{a} \qquad \overline{G}(-\frac{1}{a}) = 0 \tag{57}$$

Therefore

$$\overline{G}(w) = \frac{T}{K}(a + w^{-1})\sum_{k=1}^{\infty} \overline{g}_k w^{-(k-1)} \tag{58}$$

$$G(z) = \overline{G}\left(\frac{z-a}{1-az}\right) = \frac{T}{K}(a + \frac{1-az}{z-a})\sum_{k=1}^{\infty} \overline{g}_k \left(\frac{z-a}{1-az}\right)^{-(k-1)} \tag{59}$$

$$\therefore G(s) = \sum_{k=1}^{\infty} \overline{g}_k \frac{K}{z-a}\left(\frac{1-az}{z-a}\right)^{k-1} \tag{53}$$

Q.E.D.

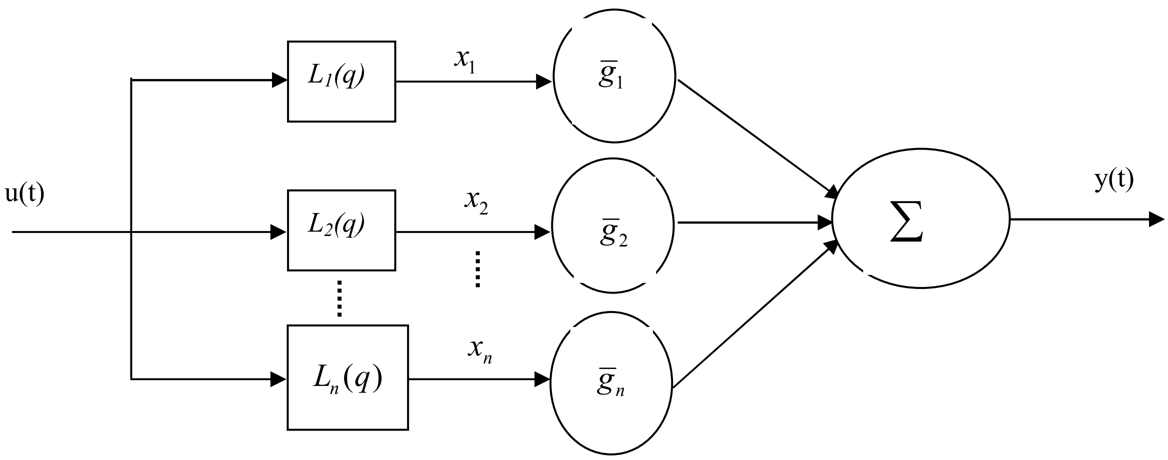
Now we can write

$$y(t) = G(q)u(t)$$
$$= \sum_{k=1}^{n} \overline{g}_k \frac{K}{q-a}\left(\frac{1-aq}{q-a}\right)^{k-1} u(t) = \sum_{k=1}^{n} \overline{g}_k L_k(q)u(t)$$

where $L_k(q), k = 1,\cdots,n$ is a series of filters. Once the original input data are filtered with $L_k(q), k = 1, \cdots, n$,

$$x_k = L_k(q)u(t), \quad k = 1, \cdots, n \tag{60}$$

The output $y(t)$ is represented as a moving average of the transformed input x_k, that is, a FIR model.



Furthermore, $x_k(t)$ can be computed recursively.

$$x_1(t) = L_1(q)u(t) = \frac{K}{q-a}u(t) = \frac{Kq^{-1}}{1-aq^{-1}}u(t)$$

$$x_1(t) - ax_1(t-1) = Ku(t-1)$$

$$x_1(t) = ax_1(t-1) + Ku(t-1)$$

$$x_2(t) = \frac{1-aq}{q-a}x_1(t) = \frac{q^{-1}-a}{1-aq^{-1}}x_1(t)$$

$$x_2(t) - ax_2(t-1) = x_1(t-1) - ax_1(t)$$

$$x_2(t) = ax_2(t-1) + x_1(t-1) - ax_1(t)$$

$$x_k(t) = ax_k(t-1) + x_{k-1}(t-1) - ax_{k-1}(t) \tag{61}$$

Recursive Filters