# PHYSICAL BASIS OF ANALOGIES

### Equivalences: Transforming Through Transformers and Gyrators

One useful way to enhance your understanding of a system involving multiple energy domains is to develop an equivalent dynamic system in a single domain of your choice. To do this, any asymmetric junction elements (transformers or gyrators) coupling the multiple domains must be eliminated, if possible. Indeed, even if the original system involves only a single energy medium, identifying an equivalent system without transformers and gyrators can be useful. In effect, it is helpful to know how an element (or even an entire subsystem) would appear to behave if it were observed through a transformer or gyrator. By loose analogy with viewing an object through a lens, we speak of transforming elements or subsystems through transformers or gyrators.

Transforming Through a Transformer

Any element or subsystem transformed through a transformer retains its dynamic character, though its parameter values change. That is, a zero junction still appears to behave as a zero junction, a capacitor still appears to behave as a capacitor, a flow source still appears to behave as a flow source, and so forth. This is true whether the elements are linear or nonlinear.

The equivalent (transformed) active source elements are related to the original source elements in proportion to the transformer coefficient, as summarized in table 5.2.

The relation for two-port asymmetric junctions (transformers and gyrators) is equally straightforward and is summarized in table 5.3. The other possible causal assignments are treated similarly.

Multi-port symmetric junctions (one and zero junctions) may also be transformed as summarized in table 5.4. Representative causal assignments are shown. The other possible causal assignments are treated similarly.

Table 5.2

Transforming active elements through a transformer.

| | Initial bond graph: | Equivalent bond graph: |
|---|---|---|
| | $\vdash\!\xrightarrow[1]{} \underset{T}{\overset{}{\mathrm{TF}}} \vdash\!\xrightarrow[2]{} S_e : e(t)$ | $\vdash\!\xrightarrow[1]{} S_e : Te(t)$ |
| Constitutive equations: | $e_1 = T\,e_2;\ e_2 = e(t)$ | $e_1 = Te(t)$ |
| | $\xrightarrow[1]{}\!\dashv \underset{1/T}{\mathrm{TF}} \xrightarrow[2]{}\!\dashv S_f : f(t)$ | $\xrightarrow[1]{}\!\dashv S_f : f(t)/T$ |
| Constitutive equations: | $f_1 = (1/T)\,f_2;\ f_2 = f(t)$ | $f_1 = f(t)/T$ |

Table 5.3

Transforming asymmetric junction elements through a transformer.

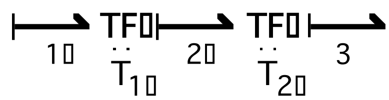
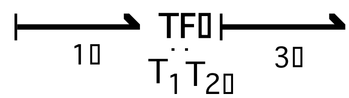
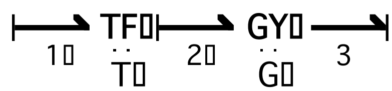
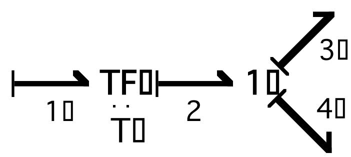
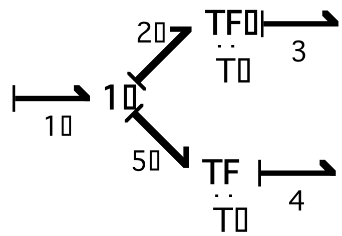
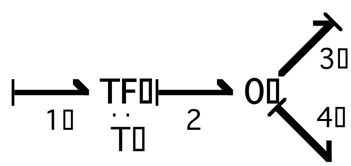
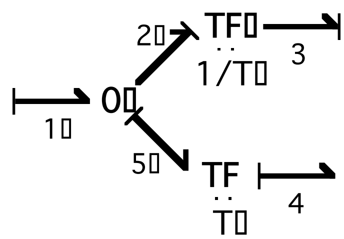
| | Initial bond graph: | Equivalent bond graph: |
|---|---|---|
| |  |  |
| Constitutive equations: | $e_1 = T_1 e_2; e_2 = T_2 e_3$<br>$f_2 = T_1 f_1; f_3 = T_2 f_2$ | $e_1 = T_1T_2 e_3$<br>$f_3 = T_1T_2 f_1$ |
| |  |  |
| Constitutive equations: | $e_1 = T e_2; e_2 = G f_3$<br>$f_2 = T f_1; e_3 = G f_2$ | $e_1 = TG f_3$<br>$e_3 = TG f_1$ |

Table 5.4

Transforming symmetric junction elements through a transformer.

| | Initial bond graph: | Equivalent bond graph: |
|---|---|---|
| |  |  |
| Constitutive equations: | $e_1 = T(e_3 + e_4)$<br>$f_3 = T f_1; f_4 = T f_1$ | $e_1 = T e_3 + T e_4$<br>$f_3 = T f_1; f_4 = T f_1$ |
| |  |  |
| Constitutive equations: | $e_1 = T e_4; e_3 = e_4$<br>$f_4 = T f_1 - f_3$ | $e_1 = T e_4; e_3 = (1/T)T e_4$<br>$f_4 = T f_1 - T(1/T) f_3$ |

For passive elements, the relation is quite different. If a passive element is linear, the parameter of the equivalent (transformed) element is related to the parameter of the original element through the *square* of the transformer coefficient. One way to remember this is to note that to obtain the equivalent relation for an active element, we had to "look through" the transformer once. In contrast, to obtain the equivalent relation for a passive element, we had to "look through" the transformer twice, and had to multiply (or divide) by the transformer coefficient on each pass. These relations are summarized in table 5.5

Table 5.5

Transforming passive elements through a transformer.

| | Initial bond graph: | Equivalent bond graph: |
|---|---|---|
| | $\underset{1}{\vdash\!\!\longrightarrow}\ \underset{T}{\mathrm{TF}}\ \underset{2}{\vdash\!\!\longrightarrow}\ \mathrm{C{:}C}$ | $\underset{1}{\vdash\!\!\longrightarrow}\ \mathrm{C{:}C/T^2}$ |
| Constitutive equations: | $e_1 = T\, e_2;\ \frac{de_2}{dt} = \frac{1}{C} f_2;\ f_2 = T\, f_1$ | $\frac{de_1}{dt} = \frac{T^2}{C} f_1$ |
| | $\underset{1}{\longrightarrow\!\!\mid}\ \underset{1/T}{\mathrm{TF}}\ \underset{2}{\longrightarrow\!\!\mid}\ \mathrm{I{:}I}$ | $\underset{1}{\longrightarrow\!\!\mid}\ \mathrm{I{:}T^2 I}$ |
| Constitutive equations: | $f_1 = \frac{1}{T} f_2;\ \frac{df_2}{dt} = \frac{1}{I} e_2;\ e_2 = \frac{1}{T} e_1$ | $\frac{df_1}{dt} = \frac{1}{T^2 I} e_1$ |
| | $\underset{1}{\vdash\!\!\longrightarrow}\ \underset{T}{\mathrm{TF}}\ \underset{2}{\vdash\!\!\longrightarrow}\ \mathrm{R{:}R}$ | $\underset{1}{\vdash\!\!\longrightarrow}\ \mathrm{R{:}T^2 R}$ |
| Constitutive equations: | $e_1 = T\, e_2;\ e_2 = R\, f_2;\ f_2 = T\, f_1$ | $e_1 = T^2 R\, f_1$ |
| | $\underset{1}{\longrightarrow\!\!\mid}\ \underset{1/T}{\mathrm{TF}}\ \underset{2}{\longrightarrow\!\!\mid}\ \mathrm{R{:}G}$ | $\underset{1}{\longrightarrow\!\!\mid}\ \mathrm{R{:}G/T^2}$ |
| Constitutive equations: | $f_1 = \frac{1}{T} f_2;\ f_2 = G\, e_2;\ e_2 = \frac{1}{T} e_1$ | $f_1 = \frac{G}{T^2} e_1$ |

## Transforming Through a Gyrator

Any element or subsystem transformed through a gyrator takes on the dynamic character of the *dual* element, and in addition its parameter values may change. That is, a zero junction appears to behave as a one junction, a capacitor appears to behave as an inertia, a flow source appears to behave as an effort source, and so forth. This is true whether the elements are linear or nonlinear.

Table 5.6

Transforming active elements through a gyrator.

| | Initial bond graph: | Equivalent bond graph: |
|---|---|---|
| | $\vdash\xrightarrow[1]{} \underset{G}{\ddot{GY}} \xrightarrow[2]{}\!\mid S_f : f(t)$ | $\vdash\xrightarrow[1]{} S_e : Gf(t)$ |
| Constitutive equations: | $e_1 = G\,f_2;\ f_2 = f(t)$ | $e_1 = Gf(t)$ |
| | $\xrightarrow[1]{}\!\mid \underset{1/G}{\ddot{GY}} \vdash\xrightarrow[2]{} S_e : e(t)$ | $\xrightarrow[1]{}\!\mid S_f : e(t)/G$ |
| Constitutive equations: | $f_1 = (1/G)\,e_2;\ e_2 = e(t)$ | $f_1 = e(t)/G$ |

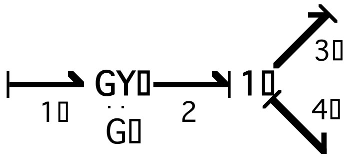
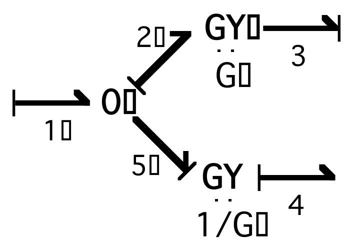
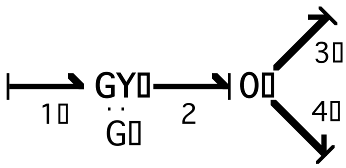
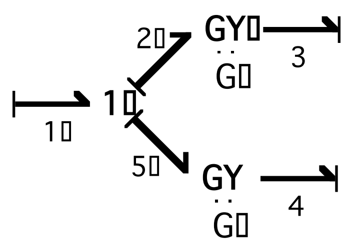
Table 5.7

Transforming asymmetric junction elements through a gyrator.

| | Initial bond graph: | Equivalent bond graph: |
|---|---|---|
| | $\vdash\xrightarrow[1]{} \underset{G}{\ddot{GY}} \xrightarrow[2]{}\!\mid \underset{1/T}{\ddot{TF}} \xrightarrow[3]{}\!\mid$ | $\vdash\xrightarrow[1]{} \underset{G/T}{\ddot{GY}} \xrightarrow[3]{}\!\mid$ |
| Constitutive equations: | $e_1 = G\,f_2;\ f_2 = (1/T)\,f_3$<br>$e_2 = G\,f_1;\ e_3 = (1/T)\,e_2$ | $e_1 = (G/T)\,f_3$<br>$f_3 = (G/T)\,f_1$ |
| | $\vdash\xrightarrow[1]{} \underset{G_1}{\ddot{GY}} \xrightarrow[2]{}\!\mid \underset{1/G_2}{\ddot{GY}} \vdash\xrightarrow[3]{}$ | $\vdash\xrightarrow[1]{} \underset{G_1/G_2}{\ddot{TF}} \vdash\xrightarrow[3]{}$ |
| Constitutive equations: | $e_1 = G_1\,f_2;\ f_2 = (1/G_2)\,e_3$<br>$e_2 = G_1\,f_1;\ f_3 = (1/G_2)\,e_2$ | $e_1 = (G_1/G_2)\,e_3$<br>$f_3 = (G_1/G_2)\,f_1$ |

Table 5.8

Transforming symmetric junction elements through a gyrator.

| | Initial bond graph: | Equivalent bond graph: |
|---|---|---|
| |   |   |
| Constitutive equations: | $e_1 = G f_3; f_4 = f_3$<br>$e_3 = G f_1 - e_4$ | $e_1 = G f_3; f_4 = (1/G)G f_3$<br>$e_3 = G(f_1 - (1/G)e_4)$ |
| |   |   |
| Constitutive equations: | $e_1 = G(f_3 + f_4)$<br>$e_3 = G f_1; e_4 = G f_1$ | $e_1 = G f_3 + G f_4$<br>$e_3 = G f_1; e_4 = G f_1$ |

The equivalent (transformed) active source elements are related to the original source elements in proportion to the gyrator coefficient, as summarized in table 5.6.

The relation for two-port asymmetric junctions (transformers and gyrators) is summarized in table 5.7. The other possible causal assignments are treated similarly.

Multi-port symmetric junctions (one and zero junctions) may also be transformed as summarized in table 5.8. Representative causal assignments are shown. The other possible causal assignments are treated similarly.

As with transforming through a transformer, if a linear passive element is transformed through a gyrator, the parameter of the equivalent element is related to the parameter of the original element through the *square* of the gyrator coefficient. These relations are summarized in table 5.9

Table 5.9

Transforming passive elements through a gyrator.

| | Initial bond graph: | Equivalent bond graph: |
|---|---|---|
| | $\vdash\!\xrightarrow[1]{} \underset{G}{\mathrm{GY}} \xrightarrow[2]{}\!\mid \mathrm{I:I}$ | $\vdash\!\xrightarrow[1]{} \mathrm{C:I/G^2}$ |
| Constitutive equations: | $e_1 = G f_2; \frac{df_2}{dt} = \frac{1}{I} e_2; e_2 = G f_1$ | $\frac{de_1}{dt} = \frac{G^2}{I} f_1$ |
| | $\xrightarrow[1]{}\!\mid \underset{1/G}{\mathrm{GY}} \vdash\!\xrightarrow[2]{} \mathrm{C:C}$ | $\xrightarrow[1]{}\!\mid \mathrm{I:G^2C}$ |
| Constitutive equations: | $f_1 = \frac{1}{G} e_2; \frac{de_2}{dt} = \frac{1}{C} f_2; f_2 = \frac{1}{G} e_1$ | $\frac{df_1}{dt} = \frac{1}{G^2C} e_1$ |
| | $\vdash\!\xrightarrow[1]{} \underset{G}{\mathrm{GY}} \xrightarrow[2]{}\!\mid \mathrm{R:G_r}$ | $\vdash\!\xrightarrow[1]{} \mathrm{R:G^2/G_r}$ |
| Constitutive equations: | $e_1 = G f_2; f_2 = \frac{e_2}{G_r}; e_2 = G f_1$ | $e_1 = \frac{G^2}{G_r} f_1$ |
| | $\xrightarrow[1]{}\!\mid \underset{1/G}{\mathrm{GY}} \vdash\!\xrightarrow[2]{} \mathrm{R:R}$ | $\xrightarrow[1]{}\!\mid \mathrm{R:G_r/G^2}$ |
| Constitutive equations: | $f_1 = \frac{1}{G} e_2; e_2 = R f_2; f_2 = \frac{1}{G} e_1$ | $f_1 = \frac{R}{G^2} e_1$ |

These tables are not intended to be memorized; they are presented primarily to demonstrate the various equivalences. Summarizing: transforming an element through a transformer may change its parameters but not its form. Transforming an element through a gyrator changes the element to its dual, and may also change its parameter values. The parameters of passive linear elements are scaled by the square of the transformer or gyrator parameter. The constitutive equations of other linear elements are changed in proportion to the first power of the parameter.

These equivalences permit entire subsystems within a bond graph to be transformed through a transformer or gyrator, thereby enhancing insight into the consequences of energetic interaction and simplifying the graph and the subsequent process of deriving equations . Beware, however, that the physical meaning of the quantities represented on the graph may become obscured.

The fact that transforming through a gyrator dualizes an element has some intriguing consequences. A transformer is, in a sense, a superfluous element, as it could always be represented by a pair of gyrators. By a similar argument, one of each pair of the other dual elements (0 and 1, C and I, $S_e$

and $S_f$) could always be replaced by the other transformed through a gyrator. Thus any bond-graph composed of the nine primitive elements we have defined so far (R, $S_e$, $S_f$, 0, 1, C, I, GY, TF) could be constructed from a set of five primitive elements, for example: (R, $S_f$, 0, C, GY). This possibility and its implications have been explored at some length in the bond-graph literature. However, most engineers are quite comfortable with the dual concepts of inertia and elasticity, inductance and capacitance, and so forth, and it is not clear that this further abstraction would enhance understanding of the physical system.