**Causality and Model Formulation**

Causal considerations can be extremely helpful at the stage of formulating a model because they can identify the consequences of particular choices.  Consider the following example.

Two large, identical electrical capacitors, initially charged to different voltages, are to be connected together by two bars of copper as shown conceptually in the figure 4.30.  This system would be a reasonable laboratory simulation of one of the problems of switching in an electrical power system.  The copper bars are large, several square centimeters in cross sectional area.  We wish to formulate the simplest possible model which is competent to describe the events that happen as the contact is closed.

The simplest model for this system would appear to be two capacitors and to keep things simple we may assume they are ideal.  The resistance of the connector would appear to be negligible.  Ignoring surface contact phenomena, the resistance of a conductor is the product of resistivity and length divided by cross-sectional area

$$R = \rho l/A \tag{4.116}$$

where R is resistance, ρ is resistivity, l is length and A is area.  Copper is an excellent conductor; the resistivity of drawn copper is listed as 1.724 μΩ-cm.  Even if the copper bar is tens of centimeters long, its resistance will be measured in micro-ohms.
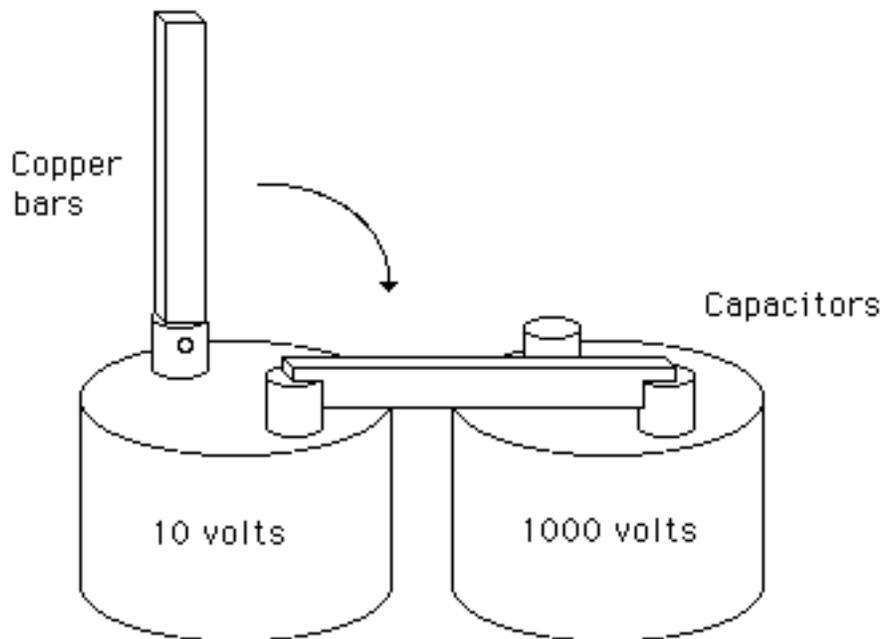


Figure 4.30:    Conceptual sketch of a capacitor and switch system.

When the switch is closed the two capacitors share a common voltage, which implies that their connection should be modeled as a zero-junction.  As a first step in thinking about this model it is useful to assign causality.  As a result, without deriving equations (admittedly a simple enough

task in this case) it is clear that the two capacitors are not independent; assign either capacitor integral causality and, because of the zero-junction, the other must have derivative causality as shown in figure 4.31.

Does this make physical sense?  Yes.  Given the capacitance of both capacitors, a knowledge of the energy stored in one is sufficient to determine its charge or voltage; the voltage is the same for both capacitors, hence the energy stored in the second capacitor can be computed.  Therefore only one state variable is needed to characterize the energetic state of this system.

$$C \longleftarrow\!\!\!| \ 0 \longrightarrow\!\!\! C$$

Figure 4.31:    Model of two capacitors connected.

Simple though this system is, there is plenty of room for confusion: before connection the charges on the two capacitors could be specified independently; afterwards, our model says they cannot.  Do we need two initial conditions or one?

The confusion arises because we have two structurally different models for the system before and after connection.  That is, different mathematical operations will be used to describe the capacitors in the two different situations.  This is easily seen by comparing the causality of the capacitors in the two situations.  A model of the unconnected capacitors would be as shown in figure 4.32.  By assumption, no current may flow from the capacitors, so this boundary condition is modeled by a flow source imposing zero flow on the capacitor.  Before connection, each of the two capacitors is independent and we are free to assign integral causality to each.

$$C \longleftarrow\!\!\!| \ S_f : 0 \qquad 0 : S_f |\!\longrightarrow C$$

Figure 4.32:    Model of the two unconnected capacitors.

There is nothing wrong with using two structurally different models to describe different operating conditions for a system, though care must be taken to ensure that the models are mutually consistent and do not contradict each other.  Is there some contradiction or inconsistency here?  No.  In the absence of charge leakage, the total charge in the system must be the same before and after connection.  Using obvious subscripts:

$$(q_1 + q_2)_{before} = (q_1 + q_2)_{after} \tag{4.117}$$

$$(Ce_1 + Ce_2)_{before} = (Ce_1 + Ce_2)_{after} = 2Ce_{after} \tag{4.118}$$

Thus the two models may be made mutually consistent by deriving the single initial condition for the model after connection from the state of the model just before connection.

$$e_{after} = \left(\frac{e_1 + e_2}{2}\right)_{before} \tag{4.119}$$

However, if we attempt to concatenate the two pieces of the model valid before connection (figure 4.32) to obtain a model for the same two capacitors after connection (figure 4.31) we

discover a "causal conflict". This "causal conflict" means that the pieces cannot just be assembled, the entire model for the coupled system has to be reformulated; new equations have to be written. In this case, the new equations require a different number of state variables than the old. But you don't have to work out the equations to see this; it is evident from the change of causal form for one of the capacitors from integral causality to derivative causality.

In this case we were asked for a model to describe the events that happen as the contact is closed. But if we want to describe the events <u>at</u> switch closure, we have a problem: How do we reconcile the different descriptions in the two regimes?

The key to understanding the dynamics of an energetic system is to keep track of energy. In this linear system, energy is numerically equal to co-energy. Before contact:

$$E_p^* \text{ before} = \tfrac{1}{2} \, C(e_1{}^2 + e_2{}^2)\text{before} \tag{4.120}$$

After contact:

$$E_p^* \text{ after} = \tfrac{1}{2} \, C(e_1{}^2 + e_2{}^2)\text{after} = C\left(\frac{e_1 + e_2}{2}\right)^2{}_{\text{before}} \tag{4.121}$$

Therefore:

$$E_p^* \text{ before} - E_p^* \text{ after} = C\left(\frac{e_1}{2} - \frac{e_2}{2}\right)^2 \ge 0 \tag{4.122}$$

Energy is lost during switch closure. To model this phenomenon we must include at least one dissipative element[1], *however small.*

The point to be taken here is that by considering causality we uncover an aspect of the model which we may want to modify. It is important to stress that "causal conflicts" are not errors. They simply spell out consequences of modelling assumptions which may otherwise be obscure. Think of them as waving a banner which reads "Are you sure you really meant your model to have this property?"

*Junction Structure*

Suppose we decide to include a dissipator. The simplest model we can assemble will include two capacitors and an ideal dissipator. How should they be connected? Any current that leaves one capacitor flows through the dissipator and enters the other capacitor; this indicates a one-junction and suggests the bond graph shown in figure 4.33. Assigning causality we see that both

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[1] In practice, the dissipative phenomenon encountered on switch closure is dominated by an arc which "jumps the gap" before metal-to-metal contact is made rather than by the resistance of the metallic components.

capacitors may be given integral causality and the dissipator has conductance causality; it determines the current as a function of the voltages on the two capacitors.
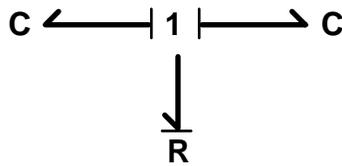


Figure 4.33:    Model of the two capacitors including dissipation.

But this graph has a serious flaw.  We expect the dissipator to determine current as a function of the potential <u>difference</u> across its terminals.  Yet the compatibility equation (Kirchhoff's voltage law) associated with the junction in figure 4.33 would be written as follows (using obvious subscripts and reading the signs and the required causal form from the graph):

$$e_R = -(e_1 + e_2) \qquad\qquad\qquad (4.123)$$

Thus the equation for the common current would depend on the negative sum of the capacitor voltages.

$$\dot{i}_R = -\frac{e_1 + e_2}{R} \qquad\qquad\qquad (4.124)$$

This is the same problem we encountered previously.  In fact, it is possible to define a sign convention for the capacitor and dissipator voltages so that equation 4.124 is correct, but it is quite counter-intuitive.  A better course of action is to revise the junction structure so that the expected potential difference is identified as shown in figure 4.34.
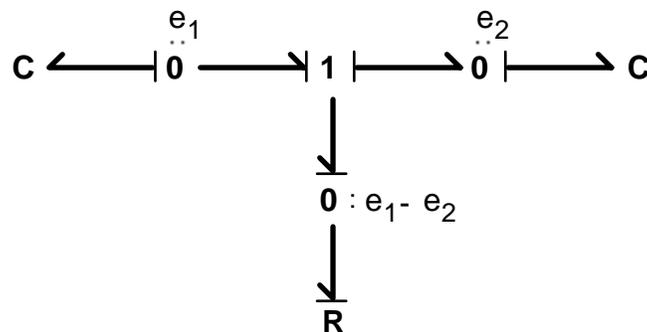


Figure 4.34:    Revised junction structure which identifies the potential difference across the dissipator.

The zero junction in the center may be eliminated, because, reading the half-arrows, its equations merely state that the efforts and flows on the associated bonds are identical.  The same is true for the zero junction on the right, but not for that on the left.  The compatibility equation for this junction states that the flows on the two associated bonds are equal and opposite; this junction cannot be eliminated.  A simplified bond graph is shown in figure 4.35.
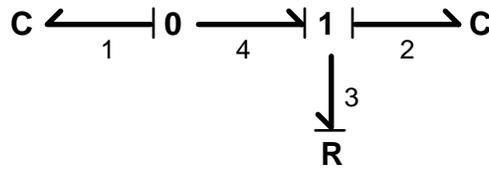
Figure 4.35:    Simplified bond graph for the two capacitors with dissipation.

The numbers written beside the bonds in figure 4.35 are not strictly necessary.  They are included solely to facilitate the following derivation.

*State Equations*

At this point, deriving state equations is straightforward.  As all elements in this system are linear we might use circuit variables (capacitor voltages in this case) for state variables.  But we may equally well use energy variables (capacitor charges in this case) and, in fact, will do so. The definition of generalized displacement yields:

$$dq_1/dt = i_1 \tag{4.125}$$

$$dq_2/dt = i_2 \tag{4.126}$$

Reading the causal strokes on the graph, $i_1$ is determined by $i_4$      which is in turn determined by $i_3$; $i_2$ is identical to $i_3$.  Reading the signs from the half arrows:

$$i_1 = -i_4 = -i_3 \tag{4.127}$$

The dissipator determines $i_3$.

$$i_3 = \frac{1}{R} \, e_3 \tag{4.128}$$

$e_3$ is determined by $e_2$ and $e_4$; $e_4$ is identical to $e_1$.  Reading the signs from the half arrows:

$$e_3 = e_4 - e_2 \tag{4.129}$$

$$e_4 = e_1 \tag{4.130}$$

The capacitors determine $e_1$ and $e_2$.

$$e_1 = \frac{q_1}{C} \tag{4.131}$$

$$e_2 = \frac{q_2}{C} \tag{4.132}$$

State equations are now obtained by substitution.

$$dq_1/dt = -\frac{1}{R}\left(\frac{q_1}{C} - \frac{q_2}{C}\right) \tag{4.133}$$

$$dq_2/dt = \frac{1}{R}\left(\frac{q_1}{C} - \frac{q_2}{C}\right) \tag{4.134}$$

These equations may be written in standard integrable matrix form $\dot{\mathbf{x}} = \mathbf{Ax}$.

$$\frac{d}{dt}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} -1/RC & 1/RC \\ 1/RC & -1/RC \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \tag{4.135}$$

Notice that the off-diagonal terms in the system matrix lack the sign asymmetry required in an oscillator. In fact, all elements in the system matrix have the same magnitude and the pattern of signs comes from the fact that $i_1 = -i_2$. It is instructive to change coordinates and use the sum and difference of the two charges as state variables. The transformation equation is as follows.

$$\begin{bmatrix} q_d \\ q_s \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \tag{4.136}$$

The inverse transformation also exists.

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}\begin{bmatrix} q_d \\ q_s \end{bmatrix} \tag{4.137}$$

From equation 4.94, the transformed state equations are:

$$\frac{d}{dt}\begin{bmatrix} q_d \\ q_s \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1/RC & 1/RC \\ 1/RC & -1/RC \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}\begin{bmatrix} q_d \\ q_s \end{bmatrix} \tag{4.138}$$

$$\frac{d}{dt}\begin{bmatrix} q_d \\ q_s \end{bmatrix} = \begin{bmatrix} -2/RC & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} q_d \\ q_s \end{bmatrix} \tag{4.139}$$

With this change of coordinates the state equations are trivial to integrate. The solutions are:

$$q_d(t) = q_d(t_o)e^{(t_o - t)2/RC} \tag{4.140}$$

$$q_s(t) = q_s(t_o) \tag{4.141}$$

Hence the second state equation merely reiterates the fact that charge is conserved in this system. As we expect from the pattern of signs in the system matrix, this system cannot exhibit oscillatory behavior.

*A Switch Behaves as a Modulated Dissipator*

As a final remark on this simple example, if we compare figures 4.32 and 4.35 we may note that the causality on capacitors is same in both cases, before and after connection. This indicates that it is possible to represent the models for both regimes as a single bond graph, shown in figure 4.36. The important addition is that the switch has been represented as a *modulated dissipator*. A suitable constitutive equation would be:

$$i_R = \frac{u}{R} \, e_R \tag{4.142}$$

where u is a modulating or control input which can assume only two real values, 1 or 0, corresponding to connected and disconnected, respectively.

$$i_R = \begin{cases} 0 \text{ if } u = 0 \\ \dfrac{e_R}{R} \text{ if } u = 1 \end{cases} \tag{4.143}$$

$$C \longleftarrow\!| 0 \longrightarrow |\!\!\rightarrow 1 |\!\longrightarrow C$$

modulating or
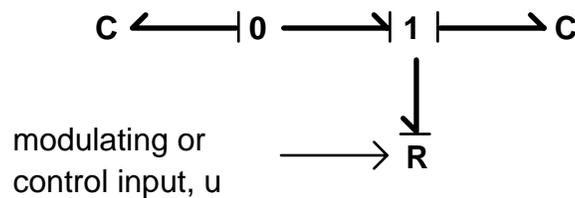control input, u $\longrightarrow \searrow R$

Figure 4.36:    Model with modulated dissipator representing connected and unconnected behavior.

Note that this modulated dissipator is subject to a strict causal constraint: it must have conductance causality, determining an output flow (current) as a function of an input effort (voltage). The inverse of equation 4.142 is not defined for u = 0.

Modulated dissipators will be encountered frequently in describing control systems where signal-level events influence power-level behavior.