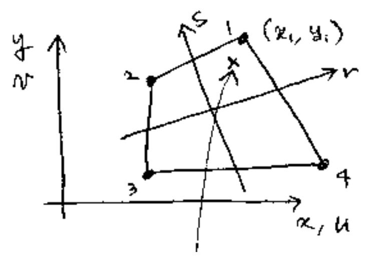
2.094 — Finite Element Analysis of Solids and Fluids	Fall '08

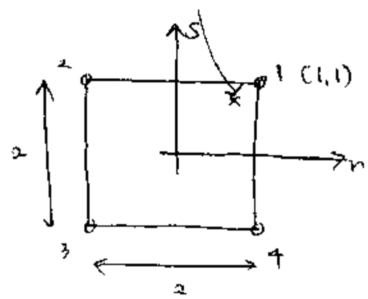
Lecture 7 - Isoparametric elements

Prof. K.J. Bathe — MIT OpenCourseWare



Reading: Sec. 5.1-5.3

We want $\boldsymbol{K} = \int_V \boldsymbol{B}^T \boldsymbol{C} \boldsymbol{B} \, dV$, $\boldsymbol{R}_B = \int_V \boldsymbol{H}^T \boldsymbol{f}^B \, dV$. Unique correspondence $(x, y) \Leftrightarrow (r, s)$



(r, s) are natural coordinate system or isoparametric coordinate system.

$$x = \sum_{i=1}^{4} h_i x_i \tag{7.1}$$

$$y = \sum_{i=1}^{4} h_i y_i \tag{7.2}$$

where

$$h_1 = \frac{1}{4}(1 + r)(1 + s) \tag{7.3}$$

$$h_2 = \frac{1}{4}(1 - r)(1 + s) \tag{7.4}$$

...

$$u(r, s) = \sum_{i=1}^{4} h_i u_i \tag{7.5}$$

$$v(r, s) = \sum_{i=1}^{4} h_i v_i \tag{7.6}$$

$$\boldsymbol{\epsilon} = \boldsymbol{B}\hat{\boldsymbol{u}} \quad \hat{\boldsymbol{u}}^T = \begin{bmatrix} u_1 & u_2 & \cdots & v_4 \end{bmatrix} \tag{7.7}$$

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} = \boldsymbol{B}\hat{\boldsymbol{u}} \tag{7.8}$$

$$\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \end{pmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix}}_{\boldsymbol{J}} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \tag{7.9}$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \boldsymbol{J}^{-1} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \end{pmatrix} \tag{7.10}$$

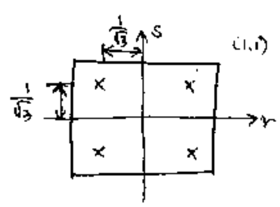
$\boldsymbol{J}$ must be non-singular which ensures that there is unique correspondence between (x, y) and (r, s). Hence,

$$\boldsymbol{K} = \int_{-1}^{1} \int_{-1}^{1} \boldsymbol{B}^T \boldsymbol{C} \boldsymbol{B} \underbrace{t \det(\boldsymbol{J})\, dr\, ds}_{d\,V} \tag{7.11}$$

$$\text{Also, } \boldsymbol{R}_B = \int_{-1}^{1} \int_{-1}^{1} \boldsymbol{H}^T \boldsymbol{f}^B \, t \det(\boldsymbol{J})\, dr\, ds \tag{7.12}$$

Numerical integration (Gauss formulae) (Ch. 5.5)

$$\boldsymbol{K} \cong t \sum_i \sum_j \boldsymbol{B}_{ij}^T \boldsymbol{C} \boldsymbol{B}_{ij} \ \det(\boldsymbol{J}_{ij}) \times (\text{weight } i,\, j) \tag{7.13}$$

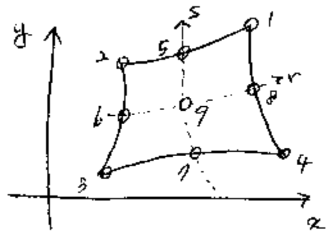


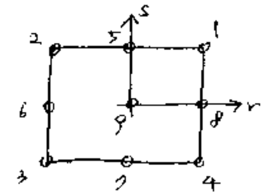
2x2 Gauss integration,

$$(i = 1, 2) \tag{7.14}$$

$$(j = 1, 2) \quad (\text{weight } i, j = 1 \text{ in this case}) \tag{7.15}$$

9-node element





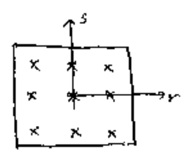
$$x = \sum_{i=1}^{9} h_i x_i \tag{7.16}$$

$$y = \sum_{i=1}^{9} h_i y_i \tag{7.17}$$

$$u = \sum_{i=1}^{9} h_i u_i \tag{7.18}$$

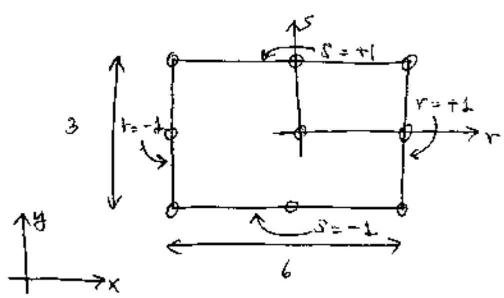
$$v = \sum_{i=1}^{9} h_i v_i \tag{7.19}$$

Use 3x3 Gauss integration



For rectangular elements, $\boldsymbol{J} = \text{const}$

Consider the following element,



Note, here we could use $h_i(x, y)$ directly.

$$\boldsymbol{J} = \begin{bmatrix} 3 \left(= \frac{6}{2}\right) & 0 \\ 0 & \frac{3}{2} \end{bmatrix} \tag{7.20}$$

Then, we can determine the number of appropriate integration points by investigating the maximum order of $\boldsymbol{B}^T \boldsymbol{C} \boldsymbol{B}$.

For a rectangular element, 3x3 Gauss integration gives exact $\boldsymbol{K}$ matrix. If the element is distorted, a $\boldsymbol{K}$ matrix which is still accurate enough will be obtained, (if high enough integration is used).

Convergence *Principle of virtual work:*

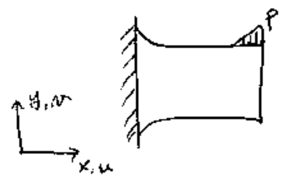
Reading: Sec. 5.5.5, 4.3

$$\int_V \overline{\boldsymbol{\epsilon}}^T \boldsymbol{C}\boldsymbol{\epsilon}\, dV = \mathcal{R}(\overline{\boldsymbol{u}}) \tag{7.21}$$

Find $\boldsymbol{u}$, solution, in V, vector space (any continuous function that satisfies boundary conditions), satisfying

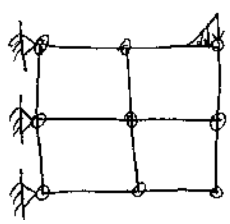
$$\int_V \overline{\boldsymbol{\epsilon}}^T \boldsymbol{C}\boldsymbol{\epsilon}\, dV = \underbrace{a(\boldsymbol{u},\boldsymbol{v})}_{\text{bilinear form}} = \underbrace{(\boldsymbol{f},\boldsymbol{v})}_{\mathcal{R}(\boldsymbol{v})} \quad \text{for all } \boldsymbol{v}\text{, an element of V.} \tag{7.22}$$

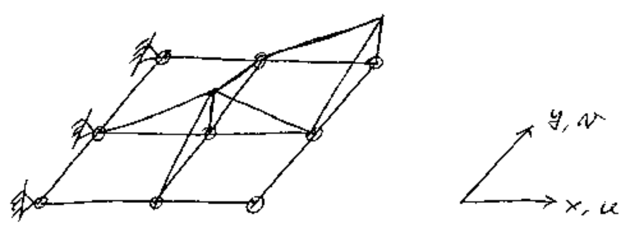
Example:



Finite Element problem Find $\boldsymbol{u}_h \in \mathrm{V}_h$, where V_h is F.E. vector space such that

$$a(\boldsymbol{u}_h,\boldsymbol{v}_h) = (\boldsymbol{f},\boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \mathrm{V}_h \tag{7.23}$$





Size of $\mathrm{V}_h \Rightarrow$ # of independent DOFs (here it's 12).

Note:

$$\underbrace{a(\boldsymbol{w},\boldsymbol{w})}_{\text{2x (strain energy when imposing } \boldsymbol{w})} > 0 \text{ for } \boldsymbol{w} \in \mathrm{V} \quad (\boldsymbol{w} \neq \boldsymbol{0})$$

Also,

$$a(\boldsymbol{w}_h,\boldsymbol{w}_h) > 0 \text{ for } \boldsymbol{w}_h \in \mathrm{V}_h \quad (\mathrm{V}_h \subset \mathrm{V},\ w_h \neq 0)$$

Property I Define: $\boldsymbol{e}_h = \boldsymbol{u} - \boldsymbol{u}_h$.

$$\text{From (7.22), } a(\boldsymbol{u}, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \tag{7.24}$$

$$\text{From (7.23), } a(\boldsymbol{u}_h, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \tag{7.25}$$

Hence,

$$a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) = 0 \tag{7.26}$$

$$a(\boldsymbol{e}_h, \boldsymbol{v}_h) = 0 \tag{7.27}$$

(error is orthogonal in that sense to all $\boldsymbol{v}_h$ in F.E. space).

Property II

$$\boxed{a(\boldsymbol{u}_h, \boldsymbol{u}_h) \leq a(\boldsymbol{u}, \boldsymbol{u})} \tag{7.28}$$

Proof:

$$a(\boldsymbol{u}, \boldsymbol{u}) = a(\boldsymbol{u}_h + \boldsymbol{e}_h, \boldsymbol{u}_h + \boldsymbol{e}_h) \tag{7.29}$$

$$= a(\boldsymbol{u}_h, \boldsymbol{u}_h) + \underbrace{2a(\boldsymbol{u}_h, \boldsymbol{e}_h)}_{0 \text{ by Prop. I}} + \underbrace{a(\boldsymbol{e}_h, \boldsymbol{e}_h)}_{\geq 0} \tag{7.30}$$

$$\therefore a(\boldsymbol{u}, \boldsymbol{u}) \geq a(\boldsymbol{u}_h, \boldsymbol{u}_h) \tag{7.31}$$