2.094

FINITE ELEMENT ANALYSIS OF SOLIDS AND FLUIDS

SPRING 2008

Homework 4 - Solution

Instructor:	Prof. K. J. Bathe	Assigned:	02/28/2008
		Due:	03/06/2008

Problem 1 (20 points):

(a)

$$h_5(x,y)=\frac{1}{24}(3-x)(4-y^2)\,; \qquad h_6(x,y)=\frac{1}{24}(3+x)(4-y^2)$$

$$h_1(x,y)=\frac{1}{24}(3+x)(2+y)-\frac{1}{2}h_6(x,y)\,; \qquad h_2(x,y)=\frac{1}{24}(3-x)(2+y)-\frac{1}{2}h_5(x,y)$$

$$h_3(x,y)=\frac{1}{24}(3-x)(2-y)-\frac{1}{2}h_5(x,y)\,; \qquad h_4(x,y)=\frac{1}{24}(3+x)(2-y)-\frac{1}{2}h_6(x,y)$$

(b) Note that above interpolation functions satisfy $\sum_{i=1}^{6} h_i(x,y)=1$

* Rigid body translation in x-direction

$$u_1=u_2=\cdots=u_6=2.0 \;\rightarrow\; u(x,y)=\sum_{i=1}^{6}h_i(x,y)u_i=2.0\sum_{i=1}^{6}h_i(x,y)=2.0$$

$$\rightarrow\; \underline{\varepsilon}^T=\begin{bmatrix}\varepsilon_{xx} & \varepsilon_{yy} & \gamma_{xy}\end{bmatrix}=\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\end{bmatrix}=\underline{0}$$

* Rigid body translation in y-direction

$$v_1=v_2=\cdots=v_6=2.0 \;\rightarrow\; v(x,y)=\sum_{i=1}^{6}h_i(x,y)v_i=2.0\sum_{i=1}^{6}h_i(x,y)=2.0$$

$$\rightarrow\; \underline{\varepsilon}^T=\begin{bmatrix}\varepsilon_{xx} & \varepsilon_{yy} & \gamma_{xy}\end{bmatrix}=\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\end{bmatrix}=\underline{0}$$

* Rigid body rotation by $60°$

Here we use: $u_i = -\theta \cdot y_i$ and $v_i = \theta \cdot x_i$ ($\theta = 60°$)

Hence $u(x,y) = \sum_{i=1}^{6} h_i(x,y)\{-\theta \cdot y_i\} = -\theta \cdot y$, $v(x,y) = \sum_{i=1}^{6} h_i(x,y)\{\theta \cdot x_i\} = \theta \cdot x$

which corresponds to the rigid body rotation. Also

$$\underline{\varepsilon}^T = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{yy} & \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} = \underline{0}$$

Note: The strains used in the class up to now correspond to infinitesimally small displacements and strains. Therefore we need to use the kinematics corresponding to these assumptions meaning $\cos\theta = 1$ and $\sin\theta = \theta$.

Later on we will allow for large deformations where we need to use $\cos\theta$ and $\sin\theta$, but then we also need to use the large strain theory.

(c)

$$\underline{H} = \begin{bmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & h_1 & h_2 & h_3 & h_4 & h_5 & h_6 \end{bmatrix}$$

$$\underline{H}^S = \underline{H}\big|_{x=-3} = \begin{bmatrix} 0 & \frac{1}{8}y(y+2) & \frac{1}{8}y(y-2) & 0 & \frac{1}{4}(4-y^2) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8}y(y+2) & \frac{1}{8}y(y-2) & 0 & \frac{1}{4}(4-y^2) & 0 \end{bmatrix}$$

$$\underline{f}^S = \begin{bmatrix} P \\ 0 \end{bmatrix}$$

$$\underline{R}_S = \int_S \left(\underline{H}^S\right)^T \underline{f}^S dS = \int_{-2}^{2} \left(\underline{H}^S\right)^T \underline{f}^S (0.1) dy$$

$$\underline{R}_S{}^T = (0.1P)\begin{bmatrix} 0 & \frac{2}{3} & \frac{2}{3} & 0 & \frac{8}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem 2 (20 points):

(a)

$$x = \sum_{i=1}^{4} h_i(r,s)x_i = \frac{1}{4}(3 + 13r + s + 3rs)$$

$$y = \sum_{i=1}^{4} h_i(r,s)y_i = (1 + 3s + rs)$$

Therefore,

$$\underline{J} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 13 + 3s & 4s \\ 1 + 3r & 12 + 4r \end{bmatrix}$$

(b)

$$\underline{J}^{-1} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{bmatrix} = \frac{1}{39 + 13r + 8s}\begin{bmatrix} 12 + 4r & -4s \\ -1 - 3r & 13 + 3s \end{bmatrix}$$

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \varepsilon_{\theta\theta} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{u}{(x+7)} \end{bmatrix} = \underline{B}\hat{\underline{u}}$$

Therefore,

$$\underline{B}_{u_1} = \begin{bmatrix} \dfrac{\partial h_1}{\partial x} \\ 0 \\ \dfrac{\partial h_1}{\partial y} \\ \dfrac{h_1}{x+7} \end{bmatrix}$$

where

$$\frac{\partial h_1}{\partial x} = \frac{\partial h_1}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial h_1}{\partial s}\frac{\partial s}{\partial x} = \frac{1+s}{4} \bullet \frac{12+4r}{39+13r+8s} - \frac{1+r}{4} \bullet \frac{4s}{39+13r+8s} = \frac{3+r+2s}{39+13r+8s}$$

$$\frac{\partial h_1}{\partial y} = \frac{\partial h_1}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial h_1}{\partial s}\frac{\partial s}{\partial y} = -\frac{1+s}{4} \bullet \frac{1+3r}{39+13r+8s} + \frac{1+r}{4} \bullet \frac{13+3s}{39+13r+8s} = \frac{6+5r+s}{2(39+13r+8s)}$$

$$\frac{h_1}{x+7} = \frac{(1+r)(1+s)}{31+13r+s+3rs}$$