2.092/2.093

FINITE ELEMENT ANALYSIS OF SOLIDS AND FLUIDS I

FALL 2009

Quiz #2-solution

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Problem 1 (10 points):

$$h_i^s = \frac{1}{4}\left(1-\frac{x}{2}\right)(1-y)\,;\quad h_i^f = \frac{1}{4}\left(1-\frac{x}{2}\right)\left(1+\frac{y}{2}\right).$$

$$h_{i,x}^s = -\frac{1}{8}(1-y)\,;\quad h_{i,y}^s = -\frac{1}{4}\left(1-\frac{x}{2}\right).$$

$$h_{i,x}^f = -\frac{1}{8}\left(1+\frac{y}{2}\right);\quad h_{i,y}^f = \frac{1}{8}\left(1-\frac{x}{2}\right)$$

$$\underline{H}^{(1)} = \begin{bmatrix} h_i^s & 0 \\ 0 & h_i^s \end{bmatrix};\quad \underline{H}^{(2)} = \begin{bmatrix} h_i^f & 0 \\ 0 & h_i^f \end{bmatrix}.$$

$$\underline{U}^T = \begin{bmatrix} u_i & v_i \end{bmatrix}.$$

$$\underline{B}^{(1)} = \begin{bmatrix} h_{i,x}^s & 0 \\ 0 & h_{i,y}^s \\ h_{i,y}^s & h_{i,x}^s \end{bmatrix};\quad \underline{B}^{(2)} = \begin{bmatrix} h_{i,x}^f & h_{i,y}^f \end{bmatrix}.$$

$$\underline{M}^{(1)} = \int_{V^{(1)}} \rho_s \underline{H}^{(1)T}\underline{H}^{(1)} dV^{(1)} = \rho_s \int_{-1}^{1}\int_{-2}^{2} \underline{H}^{(1)T}\underline{H}^{(1)} dxdy$$

$$\underline{M}^{(2)}=\int_{V^{(2)}}\rho_f\,\underline{H}^{(2)T}\underline{H}^{(2)}dV^{(2)}=\rho_f\int_{-2}^{2}\int_{-2}^{2}\underline{H}^{(2)T}\underline{H}^{(2)}dxdy$$

$$\underline{K}^{(1)}=\int_{V^{(1)}}\underline{B}^{(1)T}\underline{C}\underline{B}^{(1)}dV^{(1)}=\int_{-1}^{1}\int_{-2}^{2}\underline{B}^{(1)T}\underline{C}\underline{B}^{(1)}dxdy$$

$$\underline{K}^{(2)}=\int_{V^{(2)}}\underline{B}^{(2)T}\beta\underline{B}^{(2)}dV^{(2)}=\int_{-2}^{2}\int_{-2}^{2}\underline{B}^{(2)T}\beta\underline{B}^{(2)}dxdy$$

$$\underline{M}=\underline{M}^{(1)}+\underline{M}^{(2)};\quad \underline{K}=\underline{K}^{(1)}+\underline{K}^{(2)}.$$

Problem 2 (10 points):

(a) "Remove Clamps" to eliminate u_1 and u_3 prior to the use of the central difference method.

$$\begin{bmatrix}0&0&0\\0&2&0\\0&0&0\end{bmatrix}\begin{bmatrix}\ddot{U}_1\\\ddot{U}_2\\\ddot{U}_3\end{bmatrix}+\begin{bmatrix}2&-1&0\\-1&4&-1\\0&-1&2\end{bmatrix}\begin{bmatrix}U_1\\U_2\\U_3\end{bmatrix}=\begin{bmatrix}0\\R_2(t)\\0\end{bmatrix}$$

$$\begin{bmatrix}0&0&0\\0&2&0\\0&0&0\end{bmatrix}\begin{bmatrix}\ddot{U}_1\\\ddot{U}_2\\\ddot{U}_3\end{bmatrix}+\begin{bmatrix}2&-1&0\\0&3.5&-1\\0&-1&2\end{bmatrix}\begin{bmatrix}U_1\\U_2\\U_3\end{bmatrix}=\begin{bmatrix}0\\R_2(t)\\0\end{bmatrix}$$

$$\begin{bmatrix}0&0&0\\0&2&0\\0&0&0\end{bmatrix}\begin{bmatrix}\ddot{U}_1\\\ddot{U}_2\\\ddot{U}_3\end{bmatrix}+\begin{bmatrix}2&-1&0\\0&3&0\\0&-1&2\end{bmatrix}\begin{bmatrix}U_1\\U_2\\U_3\end{bmatrix}=\begin{bmatrix}0\\R_2(t)\\0\end{bmatrix}$$

$$2\ddot{U}_2+3U_2=R_2(t)$$

Using the central difference method,

$$2\,{}^{t}\ddot{U}_2+3\,{}^{t}U_2={}^{t}R_2 \qquad (1)$$

$$ {}^{t}\ddot{U}_2 = \frac{1}{\Delta t^2}\left({}^{t+\Delta t}U_2 - 2\,{}^{t}U_2 + {}^{t-\Delta t}U_2 \right) \qquad (2) $$

$$ {}^{t}\dot{U}_2 = \frac{1}{2\Delta t}\left({}^{t+\Delta t}U_2 - {}^{t-\Delta t}U_2 \right) \qquad (3) $$

Substitute (2) into (1)

$$ \frac{2}{\Delta t^2}\, {}^{t+\Delta t}U_2 = {}^{t}R_2 - (3 - \frac{4}{\Delta t^2})\, {}^{t}U_2 - \frac{2}{\Delta t^2}\, {}^{t-\Delta t}U_2 $$

To start the solution, we use $2\ddot{U}_2 + 3U_2 = R_2(t)$ at time t=0.

Hence, ${}^{0}\ddot{U}_2 = 0$ since U_2 and R_2 are zero at t=0.

We can obtain ${}^{-\Delta t}U_2 = 0$ using Eq. (2) and Eq. (3).

$$ 0 = {}^{0}\ddot{U}_2 = \frac{1}{\Delta t^2}\left({}^{\Delta t}U_2 - 2\,{}^{0}U_2 + {}^{-\Delta t}U_2 \right) $$

$$ 0 = {}^{0}\dot{U} = \frac{1}{2\Delta t}\left({}^{\Delta t}U_2 - {}^{-\Delta t}U_2 \right) $$

Determine ${}^{t+\Delta t}U_1$ and ${}^{t+\Delta t}U_3$ by $\begin{bmatrix} {}^{t+\Delta t}U_1 \\ {}^{t+\Delta t}U_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} {}^{t+\Delta t}U_2$.

(b) $\Delta t_{cr} = \frac{2}{\omega} = 2\sqrt{\frac{2}{3}} = 1.633$

For stability, Δt should be smaller than Δt_{cr}. The frequency content of the load can be roughly approximated by $\hat{\omega} = \frac{2\pi}{80}$ where $\hat{T} = 80$ so that $\frac{\hat{\omega}}{\omega} = 0.0641$. Because this value is quite close to zero, we can assume the response to be almost static. Therefore, $\Delta t = 1.6$ is a well selected time step for the stability and the accuracy.

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