2.092/2.093

FINITE ELEMENT ANALYSIS OF SOLIDS AND FLUIDS I

FALL 2009

Homework 8-solution

Instructor:	Prof. K. J. Bathe	Assigned:	Session 23
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Problem 1 (20 points):

a) static correction

$$\Delta \underline{R} = \underline{R} - \sum_{i=1}^{p} \left(\underline{M} \underline{\phi}_i r_i \right)$$

where p=1.

Therefore $\Delta \underline{R} = \underline{R} - \underline{M} \underline{\phi}_1 r_1 = \begin{bmatrix} 10 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0.3029 \\ 0.6739 \end{bmatrix} 3.029 = \begin{bmatrix} 9.0825 \\ -4.0825 \end{bmatrix}$

Calculate $\underline{K} \Delta \underline{U}^s = \Delta \underline{R}$ using Gauss elimination.

$$\Delta \underline{U}^s = \begin{bmatrix} 2.1498 \\ -0.4832 \end{bmatrix} \text{ and } \underline{U} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0.3029 \\ 0.6739 \end{bmatrix} 1.7062(1\text{-}\cos\sqrt{1.7753}t) + \begin{bmatrix} 2.1498 \\ -0.4832 \end{bmatrix}.$$

b)

$$\underline{K} = \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}, \ \underline{M} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \ \underline{R} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

${}^0\underline{U} = 0$; ${}^0\dot{\underline{U}} = 0$

Considering the eigenproblem, $\underline{K}\underline{\phi} = \omega^2 \underline{M} \underline{\phi}$

$\omega_1^2 = 1.7753$, $\underline{\phi}_1 = \begin{bmatrix} 0.3029 \\ 0.6739 \end{bmatrix}$ Note: $\underline{\phi}_i^T \underline{M} \underline{\phi}_j = \delta_{ij}$, $\underline{\phi}_i^T \underline{K} \underline{\phi}_j = \omega_i^2 \delta_{ij}$

$$\omega_2^2 = 4.2247\,,\;\; \underline{\phi}_2 = \begin{bmatrix} -0.9531 \\ 0.2142 \end{bmatrix}$$

Using $\underline{\text{U}}=\underline{\Phi}\underline{\text{X}}$ where $\underline{\Phi} = \begin{bmatrix} \underline{\phi}_1 & \underline{\phi}_2 \end{bmatrix}$

$$\ddot{\underline{\text{X}}}+\begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}\underline{\text{X}}=\underline{\Phi}^{\text{T}}\begin{bmatrix} 10 \\ 0 \end{bmatrix}=\begin{bmatrix} 3.029 \\ -9.531 \end{bmatrix} \qquad (1)$$

The generalized solution for (1) is

$$\underline{\text{X}}=\begin{bmatrix} \text{x}_1 \\ \text{x}_2 \end{bmatrix}=\begin{bmatrix} \text{Asin}\omega_1\text{t+Bcos}\omega_1\text{t+}\dfrac{3.029}{\omega_1^2} \\ \text{Asin}\omega_2\text{t+Bcos}\omega_2\text{t+}\dfrac{-9.531}{\omega_2^2} \end{bmatrix}=\begin{bmatrix} \text{Asin}\omega_1\text{t+Bcos}\omega_1\text{t+1.7062} \\ \text{Asin}\omega_2\text{t+Bcos}\omega_2\text{t-2.2560} \end{bmatrix}$$

From ${}^0\underline{\text{U}}={}^0\dot{\underline{\text{U}}}=0$, $\underline{\text{X}}=0$ and $\dot{\underline{\text{X}}}=0$

Using these initial conditions,

$$\begin{bmatrix} \text{x}_1 \\ \text{x}_2 \end{bmatrix}=\begin{bmatrix} \text{1.7062(1-cos}\omega_1\text{t)} \\ \text{-2.2560(1-cos}\omega_2\text{t)} \end{bmatrix}=\begin{bmatrix} \text{1.7062(1-cos}\sqrt{1.7753}\text{t)} \\ \text{-2.2560(1-cos}\sqrt{4.2247}\text{t)} \end{bmatrix}$$

Therefore, $\underline{\text{U}}=\underline{\Phi}\underline{\text{X}}=\begin{bmatrix} 0.3029 & -0.9531 \\ 0.6739 & 0.2142 \end{bmatrix}\begin{bmatrix} \text{1.7062(1-cos}\sqrt{1.7753}\text{t)} \\ \text{-2.2560(1-cos}\sqrt{4.2247}\text{t)} \end{bmatrix}$

$$\underline{\text{U}}=\begin{bmatrix} \text{U}_1 \\ \text{U}_2 \end{bmatrix}=\begin{bmatrix} 0.3029 \\ 0.6739 \end{bmatrix}\text{1.7062(1-cos}\sqrt{1.7753}\text{t)}+\begin{bmatrix} -0.9531 \\ 0.2142 \end{bmatrix}(\text{-2.2560})\,\text{(1-cos}\sqrt{4.2247}\text{t)}$$

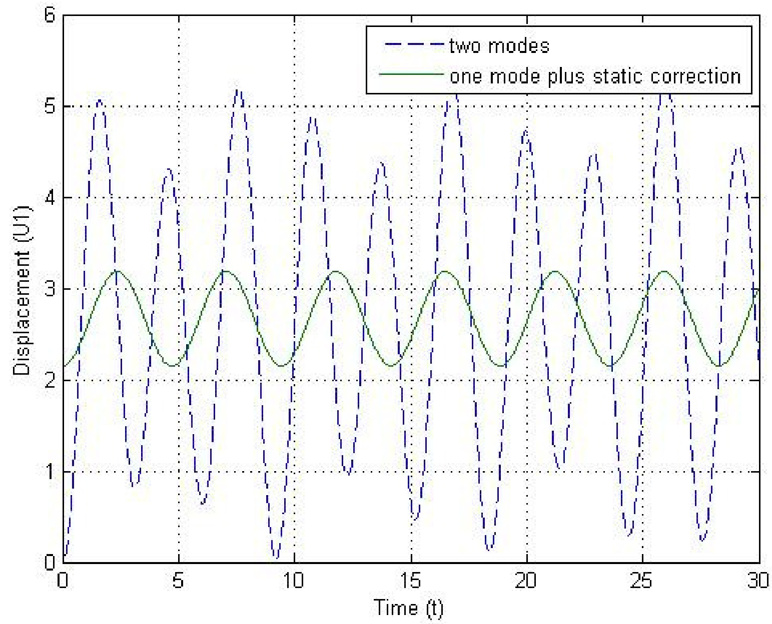

Figure 1: Comparison of the results for the displacement U_1 between (i) and (ii).

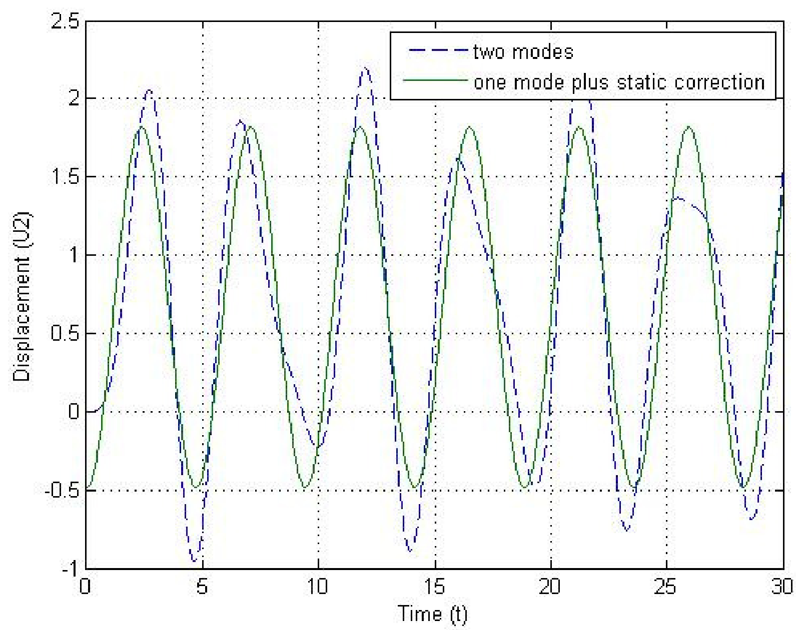

Figure 2: Comparison of the results for the displacement U_2 between (i) and (ii).

Discussion:

One mode plus static correction solution :

$$\underline{U}=\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}=\begin{bmatrix} 0.5168 \\ 1.1498 \end{bmatrix}(1\text{-}\cos\sqrt{1.7753}t)+\begin{bmatrix} 2.1498 \\ -0.4832 \end{bmatrix}$$

Two mode solution:

$$\underline{U}=\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}=\begin{bmatrix} 0.5168 \\ 1.1498 \end{bmatrix}(1\text{-}\cos\sqrt{1.7753}t)+\begin{bmatrix} 2.1502 \\ -0.4832 \end{bmatrix}(1\text{-}\cos\sqrt{4.2247}t)$$

In the one mode plus static correction solution, the static correction term shifts the displacements in such a way that the mean of the displacements is about the mean of the solution using two modes. However, here the two modes need clearly be used to obtain an accurate solution.

Problem 2 (10 points):

$$\underline{\phi}_i^T \underline{C}\underline{\phi}_j=2\omega_i\xi_i\delta_{ij} \qquad (1)$$

$$\underline{C}=\alpha\underline{M}+\beta\underline{K} \qquad (2)$$

Substitute (2) into (1)

$$\underline{\phi}_i^T\left(\alpha\underline{M}+\beta\underline{K}\right)\underline{\phi}_i=2\omega_i\xi_i$$

$$\omega_1=\sqrt{1.7753}\,,\ \omega_2=\sqrt{4.2247}\,,\ \xi_1=0.02\,,\ \xi_2=0.10$$

We obtain two equations for α and β.

$$\alpha+1.7753\beta=0.0533$$

$$\alpha+4.2247\beta=0.4111$$

$$\alpha=-0.206\,,\ \beta=0.1461$$

$$\underline{C}=-0.206\underline{M}+0.1461\underline{K}$$

Problem 3 (20 points):

a)

$$\begin{bmatrix} 4 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 4 \end{bmatrix} \underline{\phi} = \lambda \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \underline{\phi}$$

$p(\lambda)=\det\left(\underline{K} - \lambda\underline{M}\right) = -6\lambda^3 + 44\lambda^2 - 84\lambda + 40 = 0$

$\lambda_1=0.723$, $\lambda_2=2$, $\lambda_3=4.6103$

For λ_1, $\begin{bmatrix} 2.5540 & -1 & 0 \\ -1 & 1.5540 & -1.7230 \\ 0 & -1.7230 & 2.5540 \end{bmatrix} \underline{\phi}_1 = \underline{0}$

and $\underline{\phi}_1^T \underline{M} \underline{\phi}_1 = 1$

Therefore, $\underline{\phi}_1^T = \begin{bmatrix} 0.1832 & 0.4680 & 0.3157 \end{bmatrix}$

Similarly for λ_2 and λ_3 with $\underline{\phi}_2^T \underline{M} \underline{\phi}_2 = \underline{\phi}_3^T \underline{M} \underline{\phi}_3 = 1$

$\underline{\phi}_2^T = \begin{bmatrix} 0.6708 & 0 & -0.2236 \end{bmatrix}$

$\underline{\phi}_3^T = \begin{bmatrix} -0.1282 & 0.6691 & -0.7190 \end{bmatrix}$

We now show that $\underline{\phi}_i^T \underline{M} \underline{\phi}_j = \delta_{ij}$ and $\underline{\phi}_i^T \underline{K} \underline{\phi}_j = \omega_i^2 \delta_{ij}$.

$\underline{\phi}_1^T \underline{M} \underline{\phi}_2 = \underline{\phi}_2^T \underline{M} \underline{\phi}_1 = 0$; $\underline{\phi}_1^T \underline{K} \underline{\phi}_2 = \underline{\phi}_2^T \underline{K} \underline{\phi}_1 = 0$

$\underline{\phi}_1^T \underline{M} \underline{\phi}_3 = \underline{\phi}_3^T \underline{M} \underline{\phi}_1 = 0$; $\underline{\phi}_1^T \underline{K} \underline{\phi}_3 = \underline{\phi}_3^T \underline{K} \underline{\phi}_1 = 0$

$\underline{\phi}_2^T \underline{M} \underline{\phi}_3 = \underline{\phi}_3^T \underline{M} \underline{\phi}_2 = 0$; $\underline{\phi}_2^T \underline{K} \underline{\phi}_3 = \underline{\phi}_3^T \underline{K} \underline{\phi}_2 = 0$

b)

Let $\underline{x}_1^T = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ and find another another $\underline{M}$- and $\underline{K}$-orthogonal vector by inspection.

Let $\underline{x}_2^T = \begin{bmatrix} 1 & \alpha & \beta \end{bmatrix}$

then $\underline{x}_1^T \underline{M} \underline{x}_2 = [1 \quad 1 \quad 1] \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ \alpha \\ \beta \end{bmatrix} = 2 + 3\alpha + 3\beta = 0$ and

$$\underline{x}_1^T \underline{K} \underline{x}_2 = [1 \quad 1 \quad 1] \begin{bmatrix} 4 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ \alpha \\ \beta \end{bmatrix} = 3 + \alpha + 3\beta = 0 .$$

Therefore, $\alpha = \frac{1}{2}$ and $\beta = -\frac{7}{6}$.

$\underline{x}_1^T = [1 \quad 1 \quad 1]$ and $\underline{x}_2^T = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{7}{6} \end{bmatrix}$ are $\underline{M}$- and $\underline{K}$-orthogonal vectors but are not eigenvectors.

Problem 4 (20 points):

The starting vectors,

$$\underline{X}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} .$$

The relation $\underline{K}\overline{\underline{X}}_2 = \underline{M}\underline{X}_1$ gives

$$\overline{\underline{X}}_2 = \begin{bmatrix} 0.925 & 0.4 \\ 1.7 & -0.4 \\ 1.175 & -0.6 \end{bmatrix}$$

Find $\underline{K}_2$ and $\underline{M}_2$.

$$\underline{K}_2 = \underline{X}_2^T \underline{K} \underline{X}_2 = \begin{bmatrix} 10.475 & -2.2 \\ -2.2 & 2.4 \end{bmatrix}; \quad \underline{M}_2 = \underline{X}_2^T \underline{M} \underline{X}_2 = \begin{bmatrix} 14.2475 & -3.52 \\ -3.52 & 1.84 \end{bmatrix}$$

Hence,

$$\underline{\Lambda}_2 = \begin{bmatrix} 0.7267 & 0 \\ 0 & 2.0205 \end{bmatrix}; \quad \underline{Q}_2 = \begin{bmatrix} 0.2438 & 0.2714 \\ -0.0821 & 1.0118 \end{bmatrix} \text{ and } \underline{X}_2 = \begin{bmatrix} 0.1926 & 0.6558 \\ 0.4473 & 0.0567 \\ 0.3357 & -0.2882 \end{bmatrix}$$

Proceeding similarly, we obtain the following results:

$$\underline{X}_3 = \begin{bmatrix} 0.1842 & 0.6653 \\ 0.4647 & 0.0256 \\ 0.3191 & -0.2512 \end{bmatrix}; \quad \underline{\Lambda}_3 = \begin{bmatrix} 0.7231 & 0 \\ 0 & 2.0039 \end{bmatrix}$$

After two iterations we have

$$\underline{\phi}_1 \doteq \begin{bmatrix} 0.1842 \\ 0.4647 \\ 0.3191 \end{bmatrix}; \quad \lambda_1 \doteq 0.7231$$

$$\underline{\phi}_2 \doteq \begin{bmatrix} 0.6653 \\ 0.0256 \\ -0.2512 \end{bmatrix}; \quad \lambda_2 \doteq 2.0039$$