PROFESSOR: Well, OK, Professor Frey invited me to give the two lectures this week on first order equations, like that one, first order dy dt. And the lectures next week will be on second order equation. So we're looking for, you could say, formulas for the solution. We'll get as far as we can with formulas, then numerical methods. Graphical methods take over in more complicated problems.

This is a model problem. It's linear. I chose it to have constant coefficient a, and let me check the units. Always good to see the units in a problem.

So let me think of this y, as the money in a bank, or bank balance, so y as in dollars, and t, time, in years. So we're looking at the ups and downs of bank balance y. The rate of change, so the units then are dollars per year. So every term in the equation has to have the right units.

So y is in dollars, so the interest rate a is percent per year, say 6% a year. So a could be 6%-- that's dimensionless-- per year, or half a percent per month if we change. So if we change units, the constant a would change from 6 to a half. But let's stay with 6.

And then q of t represents deposits and withdrawals, so that's in dollars per year again. Has to be. So that's continuous. We think of the deposits and the interest as being computed continuously as time goes forward.

So if that's a constant-- and I'll take that case first, q equal 1-- that would mean that we're putting in, depositing $1 per year, continuously through the year. So that's the model that comes from a differential equation. A difference equation would give us finite time steps.

So I'm looking for the solution. And with constant coefficients, linear, we're going to get a formula for the solution. I could actually deal with variable interest rate for this one first order equation, but the formula becomes messy. But you can still do it.

After that point, for a second order equations like oscillation, or for a system of several equations coupled together, constant coefficients is where you can get formulas. So let's go with that case.

So how to solve that equation? Let me take first of all, a constant, constant source. So I think of q as the source term. To get one nice formula, let me take this example, ay plus 1, let's say.

How do you find y of t to solve that? And you start with some initial condition y of 0. That's the opening deposit that you make at time 0. How to solve that equation?

Well, we're looking for a solution. And solutions to linear equations have two parts. So the same will happen in linear algebra. One part is a solution to that equation, so we're just looking for one, any one, and we'll call it a particular solution. And the associated null equation, dy dt equal ay.

So this is an equation with q equals 0. That's why it's called null. And it's also called homogeneous. So more textbooks use that long word homogeneous, but I use the word null because it's shorter and because it's the same word in linear algebra.

So let me call yn the null solution, the general null solution. And y, I'm looking here for a particular solution yp, and I'm going to-- here's the key for linear equations. Let me take that off and focus on those two equations.

How does solving the null equation, which is easy to do, help me? Why can I, as I plan to do, add in yn to yp? I just add the two equations. Can I just add those two equations?

I get the derivative of yp plus yn on the left side. And I have a times yp plus yn. And that is a critical moment there when we use linearity. I had a yp a yn, and I could put them together. If it was y squared, yp squared and yn squared would not be the same as yp plus yn squared. It's the linearity that comes, and then I add the 1.

So what do I see from this? I see that yp plus yn also solves my equation. So the whole family of solutions is 1 yp plus any yn. And why do I say any yn? Because when I find one, I find more.

The solutions to this equation are yn could be e to the at, because the derivative of e to the at does bring down a factor a. But you see, I've left space for any multiple of e to the at. This is where that long word homogeneous comes from. It's homogeneous means I can multiply by any constant, and I still solve the equation. And of course, the key again is linear.

So now I have-- well, you could say I've done half the job. I've found yn, the general yn. And now I just have to find one yp, one solution to the equation. And with this source term, a constant, there's a nice way to find that solution. Look for a constant solution.

So certain right hand sides, and those are the like the special functions for the special source terms for differential equation, certain right hand sides-- and I'm just going to go down a list of them today. The next one on the list-- can I tell you what the next one on the list will be? y prime equal ay. I use prime for-- well, I'll write dy dt, but often I'll write y prime. dy dt equal ay plus an exponential. That'll be number two. So I'm just preparing the way for number two.

Well, actually number one, this example is the same as that exponential example with exponent s equal 0, right? If s is 0, then I have a constant. So this is a special case of that one. This is the most important source term in the whole subject.

But here we go with a constant 1. So we've got yn. And what's yp? I just looked to see. Can I think of one? And with these special functions, you can often find a solution of the same form as the source term. And in this case, that means a constant.

So if yp is a constant, this will be 0. So I just want to pick the constant that makes this thing 0. And of course, their right hand side is 0 when yp is minus 1 over a. So

I've got it. We've solved that equation, except we didn't match the initial condition yet.

Let me if you take that final step. So the general y is any multiple, any null solution, plus any one particular solution, that one. And we want to match it to y of 0 at t equals 0. So I want to take that solution. I want to find that constant, here. That's the only remaining step is find that constant. You've done it in the homework.

So at t equals 0, y of 0 is-- at t equals 0, this is C. This is the minus 1 over a. So I learn what the C has to be. And that's the final step. C is bring the 1 over a onto that side, so C will be y of 0 e to the at minus 1 over a e to at. And here we had a minus 1 over a. Well, it'll be plus 1 over a e to the at.

So now I've just put in the C, y of 0 plus 1 over a. y of 0 plus 1 over a has gone in for C. And now I have to subtract this 1 over a. Here, I see a 1 over a, so I can do it neatly.

Got a solution. We can check it, of course. At t equals 0, this disappears, and this is y of 0. And it has the form. It's a multiple of e to the at and a particular solution. So that's a good one.

Notice that to get the initial condition right, I couldn't take C to be y of 0 to get the initial condition right. To get the initial condition right, I had to get that, this minus 1 over a in there. Good for that one? Let me move to the next one, exponentials.

So again, we know that the null equation with no source has this solution e to the at. And I'm going to suppose that the a in e to the at in the null solution is different from the s in the source function, which will come up in the particular solution. So you're going to see either the st in the particular solution and an e to the at in the null solution.

And in the case when s equals a, that's called resonance, the two exponents are the same, and the formula changes a little. Let's leave that case for later.

How do I solve this? I'm looking for a particular solution because I know the null solutions. How am I going to get a particular solution of this equation? Fundamental observation, the key point is it's going to be a multiple of e to the st.

If an exponential goes in, then that will be an exponential. Its derivative will be an exponential. I'll have e to the st's everywhere. And I can get the number right.

So I'm looking for y try. So I'll put try, knowing it's going to work, as some number times e to the st. So this would be like the exponential response. Response, do you know that word response? So response is the solution. The input is q, and the response is Y.

And here, the input is e to the st, and the response is a multiple of e to the st. So plug it in. The timed derivative will be Y. Taking the derivative will bring down a 1. e to the st equals aY. A aY e to the st plus 1 e to the st. Just what we hoped.

The beauty of exponentials is that when you take their derivatives, you just have more exponential. That's the key thing. That's why exponential is the most important function in this course, absolutely the most important function.

So it happened here. I can cancel e to the st, because every term has one of them. So I'm seeing that-- what am I getting for Y? Getting a very important number for Y. So I bring aY onto this side with sY. On this side I just have a 1. Maybe it's worth putting on its own board.

Y is, so Ys aY comes with a minus, and the 1, 1 over-- so Y was multiplied by s minus a. That's the right quantity to get a particular solution. And that 1 over s minus a, you see why I wanted s to be different from a. I If s equaled a in that case, in that possibility of resonance when the two exponents are the same, we would have 1 over 0, and we'd have to look somewhere else.

The name for that-- this has to have a name because it shows up all the time. The exponential response function, you could call it that. Most people would call it the transfer function.

So any constant coefficient linear equation's going to have a transfer function, easy to find. Everything easy, that's what I'm emphasizing, here. Everything's straightforward. That transfer function tells you what multiplies the exponential.

So the source was here. And the response is here, the response factor, you could say, the transfer function. Multiply by 1 over s minus a. So if s is close to a, if the input is almost at the same exponent as the natural, as the null solution, then we're going to get a big response.

So that's good. For a constant coefficient problem second order, other problems we can find that response function. It's the key function. It's the function if we have, or if we were to look at Laplace transforms, that would be the key. When you take Laplace transforms, the transfer function shows up. Then when you take inverse Laplace transforms, you have to find what function has that Laplace transform.

So did we get the-- we got the final answer then. Let me put it here. y is e to the st times this factor. So I divide by s minus a. A nice solution.

Let me also anticipate something more. An important case for e to the st is e to the i omega t. e to the st, we think about as exponential growth, exponential decay. But that's for positive s and negative s.

And all important in applications is oscillation. So coming, let me say, coming is either late today or early Wednesday will be s equal i omega, so where the source term is e to the i omega t. And alternating, so this is electrical engineers would meet it constantly from alternating voltage source, alternating current source, AC, with frequency omega, 60 cycles per second, for example.

Why don't I just deal with this now? Because it involves complex numbers. And we've got to take a little step back and prepare for that. But when we do it, we'll get not only e to the i omega t, which I brought out, but also, it's real part.

You remember the great formula with complex numbers, Euler's formula, that e to the i omega t is a combination of cosine omega t, the real part, and then the imaginary part is sine omega t. So this is looking like a complex problem. But it actually solves two real problems, cosine and sine.

Cosine and sine will be on our short list of great functions that we can deal with. But to deal with them neatly, we need a little thought about complex numbers. So OK if I leave e to the i omega t for the end of the list, here?

So I'm ready for another one, another source term. And I'm going to pick the step function. So the next example is going to be dy dt equals ay plus a step. Well, suppose I put H of t there. Suppose I put H of t. And I ask you for the solution to that guy.

So that step function, its graph is here. It's 0 for negative time, and it's 1 for positive time. So we've already solved that problem, right? Where did I solve this equation? This equation is already on that board. Because why?

Because H of t is for t positive. That's the only place we're looking. This whole problem, we're not looking at negative t. We're only looking at t from 0 forward.

And what is H of t from 0 forward? It's 1. It's a constant. So that problem, as it stands, is identical to that problem. Same thing, we have a 1.

No need to solve that again. The real example is when this function jumps up at some later time T. Now I have the function is H of t minus T. Do you see that, why the step function that jumps at time T has that formula?

Because for little t before that time, in here, this is-- what's the deal? If little t is smaller than big T, then t minus T is negative, right? If t is in here, then t minus capital T is going to be a negative number. And H of a negative number is 0.

But for t greater than capital T, this is a positive number. And H of a positive number is 1. Do you see how if you want to shift a graph, if you want the graph to shift, if you want to move the starting time, then algebraically, the way you do it is to change t to t minus the starting time. And that's what I want to do.

So physically, what's happening with this equation? So it starts with y of 0 as before. Let's think of a bank balance and then other things, too. If it's a bank balance, we put in a certain amount, y of 0. We hope. And that grew.

And then starting at time, capital T, this switch turns on. Actually, physically, step function is really often describing a switch that's turned on, now. This source term act begins to act at that time. And it acts at 1.

So at time capital T we start putting money into our account. Or taking it out, of course. If this with a minus sign, I'd be putting money in. Sorry, I would start with some money in, y of 0. I would start with money in.

Yeah, actually, tell me what's the solution to this equation that starts from y of 0? What's the solution up until the switch is turned on? What's the solution before this switch happens, this solution while this is still 0? So let's put that part of the answer down.

This is for t smaller than T. What's the answer? This is all common sense. It's coming fast, so I'm asking these questions. And when I asked that question, it's a sort of indication that you can really see the answer. You don't need to go back to the textbook for that. What have we got here? Yeah?

AUDIENCE: Is it the null solution [INAUDIBLE]?

PROFESSOR: It'll be this guy. Yeah, the particular solution will be 0. Right, the particular solution is 0 before this is on. I'm sorry, the null solution is 0, and the particular solution, well, the particular solution is a guy that starts right. I don't know. Those names were not important. And then the question is-- so it's just our initial deposit growing.

Now, all I ask, what about after time T? What about after time T? For t after time T, and hopefully, equal time T, what do you think y of t will be? Again, we want to separate in our minds the stuff that's starting from the initial condition from the stuff that's piling up because of the source.

So one part will be that guy. I haven't given the complete answer. But this is continuing to grow. And because it's linear, we're always using this neat fact that our equation is linear. We can watch things separately, and then just add them together.

So I plan to add this part, which comes from initial condition to a part that-- maybe we can guess it-- that's coming from the source. And how do we have any chance to guess it? Only because that particular source, once it's turned on, jumps to a constant 1, and we've solved the equation for a constant 1.

Let me go back here. I think our answer to this question-- so this is like just first practice with a step function, to get the hang of a step function. So I'm seeing this same y of 0 e to the at in every case, because that's what happens to the initial deposit. I'll say grow, assuming the bank's paying a positive interest rate.

And now, where did this term comes from? What did that term represent?

AUDIENCE: The money that [INAUDIBLE].

PROFESSOR: The money that, yeah?

AUDIENCE: They had each of [INAUDIBLE].

PROFESSOR: The money that came in and grew. It came in, and then it grew by itself, grew separately from that these guys. So the initial condition is growing along. And the money we put in starts growing.

Now, the point is what? That over here, it's going to look just like that. So I'm going to have a 1 over a. And I'm going to have something like that. But can you just guess what's going to go in there? When I write it down, it'll make sense.

So this term is representing what we have at time little t, later on, from the deposits we made, not the initial one, but the source, the continuing deposits. And let me write it. It's going to be a 1 over a e to the a something minus 1. It's going to look just like that guy. When I say that guy, let me point to it again-- e to the at minus 1. But it's not quite e to the at minus 1. What is it?

AUDIENCE: t minus [INAUDIBLE].

PROFESSOR: t minus capital T, because it didn't start until that time. So I'm going to leave that as, like, reasonable, sensible. Think about a step function that's turned on a capital time

T. Then it grows from that time.

Of course, mentally, I never write down a formula like that without checking at t equal to T, because that's the one important point, at t equal capital T. What is this at t equal capital T? It's 0. At t equal capital T, this is e to the 0, which is 1 minus 1 altogether 0.

And is that the right answer? At t equal capital T is 0, should I have nothing here? Yes? No? Give me a head shake. Should I have nothing at t equal capital T? I've got nothing.

e to the 0 minus 1, that's nothing? Yes, yes that's the right thing. Because at capital T, the source has just turned on, hasn't had time to build up anything, just that was the instant it turned on.

So that's a step function. A step function is a little bit of a stretch from an ordinary function, but not as much of a stretch as its derivative. In a way, this is like the highlight for today, coming up, to deal with not only a step function, but a delta function.

I guess every author and every teacher has to think am I going to let this delta function into my course or into the book? And my answer is yes. You have to do it. You should do it.

Delta functions are-- they're not true functions. As we'll see, no true function can do what a delta function does. But it's such an intuitive, fantastic model of things happening over a very, very short time. We just make that short time into 0. So we're saying with the delta function, we're going to say that something can happen in 0 time. Something can happen in 0 time.

It's a model of, you know, when a bat hits a ball. There's a very short time. Or a golf club hits a golf ball. There's a very short time interval when they're in contact. We're modeling that by 0 time, but still, the ball gets an impulse.

Normally, for 0 time, if you're doing things continuously, what you do over 0 time is no importance. But we're not doing things continuously, at all. So here we go. You've seen this guy, I think. But if you haven't, here's the time to see it.

So the delta function is the derivative of-- so I've written three important functions up here. Let me start with a continuous one. That function, the ramp is 0, and then the ramp suddenly ramps up to t.

Take its derivative. So the derivative, the slope of the ramp function is certainly 0 there. And here, the slope is 1. So the slope jumped from 0 to 1. The slope of the ramp function is the step function. Derivative of ramp equals step.

Why don't I write those words down? Derivative of ramp equals step. So there is already the step function.

In pure calculus, the step function has already got a little question mark. Because at that point, the derivative in a calculus course doesn't exist, strictly doesn't exist, because we get a different answer 0 on the left side from the answer, 1 on the right side.

We just go with that. I'm not going to worry about what is its value at that point. It's 0 up for t negative, and it's 1 for t positive. And often, I'll take it 1 for t equals 0, also. Usually, I will. That's the small problem.

Now, the bigger problem is the derivative of the-- so this is now the derivative of the step function. So what's the derivative of this step function? Well, the derivative along there is certainly 0. The derivative along here is certainly 0. But the derivative, when that jumped, the derivative, the slope was infinite.

That line is vertical. Its slope is infinite. So at that one point, you have an affinity, here, delta of 0. You could say delta of 0 is infinite. But you haven't said much, there. Infinite is too vague.

Actually, I wouldn't know if you gave me infinite or 2 times infinite. I couldn't tell the difference. So I'll put it in quotes, because it sort of gives us comfort. But it doesn't mean much.

What does mean much? Somehow that's important. Can I tell you how to work with delta functions, how to think about delta functions? It's the right way to think about delta function. So here's some comment on delta function.

Giving the values of the function, 0, and infinity, and 0, is not the best. What you can do with a delta function is you can integrate it. You can define the function by integrals.

Integrals of things are nice. Do you think in your mind when you take derivatives, as we did going left to right, we were taking derivatives. The function was getting crazy. When we go right to left, take integrals, those are smoothing. Integrals make functions smoother. They cancel noise. They smooth the function out.

So what we can do is to take the integral of the delta function. We could take it from any negative number to any positive number. And what answer would we get? What would be the right, well, the one thing people know about the delta function is-- and actually, it's the key thing-- the integral of the delta function. Again, I'm integrating the delta function from some negative number up to some positive number. And it doesn't matter where n is, because the function is 0 there and there.

But what's the answer here? Put me out of my misery. Just tell me the number I'm looking for, here, the integral of the delta function. Or maybe you haven't met it.

AUDIENCE: [INAUDIBLE].

PROFESSOR: It's? It's the one good number you could guess. It's 1. Now, why is it 1? Because if the delta function is the derivative of the step function, this should be the step function evaluated between N and P. This should be the step function, , here, minus the step function, there

And what is the step function? You have to keep it straight. Am I talking about the delta function? No, right now, I've integrated it to get H of t. So this is H of P at the positive side, minus H of N. That's what integration's about.

And what do I get? 1, because H of P, the step function here, H is 1. And here, it's

0, so I get 1. Good, that's the thing that everybody remembers about the delta function.

And now I can make sense out of 2 delta function, 2 delta of t. That could be my source. So if 2 delta of t was my source, what's the graph of 2 delta of t? Again, it's 0 infinite 0. You really can't tell from the infinity what's up, but what would be the integral of 2 delta of t, the integral of 2 delta of t or some other?

Well, let me put in the 2, here? What's the integral of 2 delta of t, would be 2H of t. Keep going. What do I get here?

AUDIENCE: 2.

PROFESSOR: It would be 2 of these guys, 2 of these, 2 of these, 2. All right? So we made sense out of the strength of the impulse, how hard the bat hit the ball. But of course, we need units in there. We have to have units. And here, the value for that unit was 2.

Now, I'm going to-- because this is really worth doing with delta functions. I didn't ask at the start have you seen them before. But they are worth seeing. And they just take a little practice. But then in the end, delta functions are way easier to work with than some complicated function that attempts to model this.

We could model that by some Gaussian curve or something. All the integrations would become impossible right away. We could model it by a step function up and a step function down. Then the integrations would be possible.

But still, we have this finite width. I could let that width go to 0 and let the height go to infinity. And what would happen? I'd get the delta function. So that's one way to create a delta function, if you like.

If you're OK with step functions, then one way to create delta is to take a big step up, step down, and then let the size of the step grow and the width of the steps shrink. Keep the area 1, because area is integral. So I keep this, that little width, times this big height equal to 1. And in the end, I get delta.

Now again, my point is that delta functions, that you really understand them. What you can legitimately do with them is integrate them. But now in later problems, we might have not a 1 or a 2, but a function in here, like cosine t, or e to the t, or q of t.

Can I practice with those? Can I put in a function f of t? I didn't leave enough space to write f of t, so I'm going to put it in here. f of t delta of t dt. And I'm going to go for the answer, there.

My question is what does that equal? You see what the question is? I got my delta function, which I only just met. And I'm multiplying it by some ordinary function.

f of t gives us no problems. Think of cosine t. Think of e to the t. What do you think is the right answer for that? What do you think is the right answer? And this tells you what the delta function is when you see this.

What do I need to know about f of t to get an answer, here? Do I need to know what f is at t equals minus 1? You could see from the way my voice asked that question that the answer is no. Why do I not care what f is at minus 1? Yeah?

AUDIENCE: Because you're multiplying by [INAUDIBLE].

PROFESSOR: Because I'm multiplying by somebody that's 0. And similarly, at f equal minus 1/2, or at f equal plus 1/3, all those f's make no difference, because they're all multiplying 0. What does make a difference? What's the key information about f that does come into the answer? f at? At just at that one point, f at?

AUDIENCE: [INAUDIBLE]

PROFESSOR: 0, f at 0 is the action. The impulse is happening. The bat's hitting the ball. So we're modeling rocket launching, here. We're launching in 0 seconds instead of a finite time.

So in other words, well, I don't know how to put this answer down other than just to write it. I guess I'm hoping you're with me in seeing that what it should be. Can I just write it?

All that matters is what f is at t equals 0, because that's where all the action is. And that f of 0, if f of 0 was the 2 that I had there a little while ago, then the answer will be 2. If f of 0 is a 1, if the answer is f of 0 times 1-- and I won't write times 1. That's ridiculous.

Now we can integrate delta functions, not just a single integral of delta, but integral of a function, a nice function times delta. And we get f of 0. So can I just, while we're on the subject of delta functions, ask you a few examples? What is the integral of e to the t delta of t dt?

AUDIENCE: It's 1.

PROFESSOR: Yeah, say it again?

AUDIENCE: It's 1.

PROFESSOR: It's 1. It's 1, right. Because e to the t, at the only point we care about, t equal 0 is 1. And what if I change that to sine t? Suppose I integrate sine t times delta of t? What do I get now? I get?

AUDIENCE: 0.

PROFESSOR: 0, right. And actually, that's totally reasonable. This is a function, which is yeah, it's an odd function. Anyway, sine, if I switch t to negative t, it goes negative. 0 is the right answer.

Let me ask you this one. What about delta of t squared? Because if we're up for a delta function, we might square it. Now we've got a high-powered function, because squaring this crazy function delta of t gives us something truly crazy. And what answer would you expect for that?

AUDIENCE: 1.

PROFESSOR: Would you expect 1? So this is like? I'm just getting intuition working, here, for delta functions. What do you think? I'm looking at the energy when I square something. OK, so we had a guess of 1. Is there another guess? Yeah?

AUDIENCE: A third?

PROFESSOR: Sorry?

AUDIENCE: 1/3.

PROFESSOR: 1/3, that's our second guess. I'm open for other guesses before I-- OK, we have a rule here for f of t. And now what is the f of t that I'm asking about in this case? It's delta of t, right? If f of t is delta of t, then that would match this. And therefore, the answer should match. Do you see what I'm shooting for, yeah?

AUDIENCE: It'd be infinity?

PROFESSOR: It'd be infinity. It would be infinity. That's delta of t squared is that's an infinite energy function. You never meet it, actually. I apologize, so so write it down there. I could erase it right away because you basically never see it. It's infinite energy. Well, I think you'd see it.

I mean, we're really going back to the days of Norbert Wiener. When I came to the math department, Norbert Wiener was still here, still alive, still walking the hallway by touching the wall and counting offices. And hard to talk to, because he always had a lot to say. And you got kind of allowed to listen. So anyway, Wiener was among the first to really use delta functions, successfully use delta functions. Anyway, this is the big one. This is the big one.

Now, so what's all that about? I guess I was trying to prepare by talking about this function prepare for the equation when that's the source. So dy equal ay plus a delta function. Let me bring that delta function in at time T.

So how do you interpret that equation? So like part of this morning's lecture is to get a first handle on an impulse. So let me write that word impulse, here. Where am I going to write it? So delta is an impulse. That's our ordinary English word for something that happens fast. And y of t is the impulse response.

And this is the most important. Well, I said e to the st was the most important. How can I have two most important examples? Well, they're a tie, let's say. e to the st is the most important ordinary function. It's the key to the whole course.

Delta of t, the impulse, is the important one because if I can solve it for a delta function, I can solve it for anything. Let's see if we can solve it for a delta function, a delta function, an impulse that starts at time T. Again, I'm just going to start writing down the solution and ask for your help what to write next. So what do you expect as a first term in the solution? So I'm starting again from y of 0.

Let's see if we can solve it by common sense. So how do I start the solution to this? Everybody sees what this equation is saying. I have an initial deposit of y of 0 that starts growing. And then at time capital T I make a deposit. At that moment, at that instant, I make a deposit of 1.

That's an instant deposit of 1. Which is, of course, what I do in reality. I take $1 to the bank. They've got it now. At time T, I give them that $1, and it starts earning interest.

So what about y of t? What do you think? What's the first term coming from y of 0? So the term coming from y of 0 will be y of 0 to start with, e to at. That takes care of the y of 0.

Now, I need something. It's like this, plus I need something that accounts for what this deposit brings. So up until time T, what do I put? So this is for t smaller than T and t bigger than T. So what goes there?

For t smaller than T, what's the benefit from the delta function? 0, didn't happen yet. For t bigger than T, what's the benefit from the delta function?

AUDIENCE: [INAUDIBLE].

PROFESSOR: For t bigger than T, well, that's right. OK, but now I've made that deposit at time capital T. Whatever's going there is whatever I'm getting from that deposit. At time capital T, I gave them $1, and they start paying interest on it. What's going to go there?

So if I gave them $1 at that initial time, so that $1 would have been part of y of 0. What did I get at a later time? e to the at.

Now I'm waiting. I'm giving them the dollar at time capital T, and it starts growing. So what do I have at a later time, for t later than capital T? What has that $1 grown into? e to the a times the-- right, it's critical. It's the elapsed time. It's the time since the deposit. Is that right? So what do I put here?

AUDIENCE: t minus capital T?

PROFESSOR: t minus capital T, good. Apologies to bug you about this, but the only way to learn this stuff from a lecture is to be part of it. So I constantly ask you, instead of just writing down a formula.

I think that looks good. So suddenly, what does this amount to at t equal capital T? Maybe I should allow t equal capital T. At t equal capital T, what do I have here?

AUDIENCE: 1.

PROFESSOR: 1. That's my $1. At t equal capital T, we've got $1. And later it's grown.

So we have now solved. We have found the impulse response. We have found the impulse response when the impulse happened at capital T. That was good going.

Now, I've given you my list of examples with the pause on the sine and cosine. I pause on the sine and cosine because one way to think about sine and cosine is to get into complex numbers. And that's really for next time.

But apart from that, we've done all the examples, so are we ready? Oh yeah, I'm going to try for the big thing, the big formula. So this is the key result of section 1.4, the solution to this equation. So I'm going back to the original equation.

And just see if we can write down a formula for the answer. So let me write the equation again. dy dt is ay plus some source. I think we can write down a formula that looks right.

And we could then actually plug it in and see, yeah, it is right. So what's going to go into this formula? We got enough examples, so now let's go for the whole thing.

So y of t, first of all, comes whatever depends on the initial condition. So how much do we have at a later time when our initial deposit was y of 0? So that's the one we've seen in every example. Every one of these things has this term growing out of y of 0. So let me put that in again. So the part that grows out of y of 0 is y of 0 e to the at. That's OK.

So that's what the initial. So our money is coming from two sources, this initial deposit, which was easy, and this continuous, over time deposit, q of t. And I have to ask you about that. That's going to be like the particular solution, the particular solution that comes from the source term. This is the solution it comes from the initial condition. So what do you think this thing looks like? I just think once we see it, we can say, yeah, that makes sense.

So now I'm saying what? If we've deposited q of t in varying amounts, maybe a constant for a while, maybe a ramp for awhile, maybe whatever, a step, how am I going to think about this? So at every time t equal to s, so I'm using little t for the time I've reached. Right? Here's t starting at 0. Now, let me use s for a time part way along.

So part way along, I input. I deposit q of s. I deposit it at time s. And then what does it do?

That money is in the bank with everybody else. It grows along with everything else. So what's the growth factor? What's the growth factor?

This is the amount I deposited at time s. And how much has it grown at time t? This is the key question, and you can answer it. It went in a time s. I'm looking at time t. What's the factor?

AUDIENCE: Is it e to the a t minus s.

PROFESSOR: e to the a t minus s. So that's the contribution to our balance at time t from our input at time s. But now, I've been inputting all the way along. s is running all the way from here to here. So finish my formula. Put me out of my misery. Or it's not misery, actually. Its success at this moment. What do I do now? I?

AUDIENCE: Integrate.

PROFESSOR: I integrate, exactly. I integrate. I integrate. So all these deposits went in. They grew that amount in the remaining time. And I integrate from 0 up to the current time t.

So you see that formula? Have a look at it. This is a general formula, and every one of those examples could be found from that formula.

If q of s was 1, that was our very first example. We could do that integration. If q of s was e to the-- anyway, we could do every one. I just want you to see that that formula makes sense.

Again, this is what grew out of the initial condition. This is what grew out of the deposit at time s. And the whole point of calculus, the whole point of learning [? 1801 ?], the integral equation part, the integrals part, is integrals just add up. This term just adds up all the later deposits, times the growth factor in the remaining time.

And as I say, if I took q of s equal 1-- the examples I gave are really the examples where you can do the integral. If q of s is e to the i omega s, I can do that integral. Actually, it's not hard to do because e to the at doesn't depend on s. I can bring an e to the at out in this case.

That formula is just worth thinking about. It's worth understanding. I didn't, like, derive it. And the book does, of course. There's something called an integrating factor. You can get at this formula systematically. I'd rather get at it and understand it.

I'm more interested in understanding what the meaning of that formula is than the algebra. Algebra is just a goal to understand, and that's what I shot for directly. And as I say, that the book also, early section of the book, uses practice in calculus.

Substitute that in to the equation. Figure out what is dy dt. And check that it works. It's worth actually looking at that end of what you need to know from calculus It's is. You should be able to plug that in for y and see that solves the equation.

Right, now I have enough time to do cosine omega t. But I don't have enough time to do it the complex way. So let me do as a final example, the equation. Let me just think. I don't know if I have enough space here.

I'm now going to do dy dt-- can I call that y prime to save a little space-- equal ay plus cosine of t. I'll take omega to be 1. Now, how could we solve that one? I'm going to solve it without complex numbers, just to see how easy or hard that is. And you'll see, actually, it's easy. But complex numbers will tell us more. So it's easy, but not totally easy.

So what did I do in the earlier example if the right hand side was a 1, a constant? I look for the solution to be a constant. If the right hand side was an exponential, I look for the solution to be an exponential.

Now, my right hand side, my source term, is a cosine. So what form of the solution am I going to look for? I naturally think, OK, look for a cosine. We could try y equals some number M cosine t.

Now, you have to see what goes wrong and how to fix it. So if I plug that in, looking for M the same way I look for capital Y earlier, I plug this in, and I get aM cosine t cosine t. But what do I get for y prime? Sine t.

And I can't match. I can make it work. I can't make a sine there magic a cosine here. So what's the solution? How do I fix it?

I better allow my solution to include some sine plus N sine t. So that's the problem with doing it, keeping things real. I'll push this through, no problem. But cosine by itself won't work. I need to have sines there, because derivatives bring out sines. So I have a combination of cosine and sine. I have a combination of cosine and sine.

So the complex method will work in one shot because e to the i omega t is a combination of cosine and sine. Or another way to say it is when I see cosine here, that's got two exponentials. That's got e to the it and e to the-- anyway. Let's go for the real one.

So I'm going to plug that into there. So I'll get sines and cosines, right? When I plug this into there, I'll have some sines and some cosines, and I'll just match the two separately. So I'm going to get two equations.

First of all, let me say what's the cosine equation? And then what's the sine equation? So when I match cosine terms, what do I have? What cosine terms do I get out of y prime, here? The derivative.

Well, the derivative of cosine is a sine. That that's not a cosine term. The derivative of sine is cosine. I think I get, if I just match cosines, I think I get an N cosine. N cosine t equal ay. How many cosines do I have from that term? ay has an M cosine t. I think I have an aM, and here I've got 1.

That was a natural step, but new to us. I'm matching the cosines. I have on the left side, with this form of the solution, the derivative will have an N cosine t. So I had N cosines, aM cosines, and 1 cosine.

Now, what if I match signs? What happens there? We're pushing more than an hour, so hang on for another five minutes, and we're there. Now, what happens if I match sines, sine t? How do I get sine t in y prime?

So take the derivative of that, and what do you have?

AUDIENCE: Minus [INAUDIBLE].

PROFESSOR: Minus M sine t. That tells me how many sine t's are in there. And on the right hand side, a times y, how many sine t's do I have from that?

AUDIENCE: You have N t's.

PROFESSOR: N, good thinking. And what about from this term? None, no sine there. So I have two equations by matching the cosines and sines. Once you see it, you could do it again. And we can solve those equations, two ordinary, very simple equations for M and N.

Let's see if I make space. Why don't I do it here, so you can see it. So how do I solve those two equations? Well, this equation gives me-- easy-- gives me M as minus aN. So I'll just put that in for N.

So I have N equals aM. But M is minus aN. I think I've got minus a squared N plus that 1. All I did was solve the equation, just by common sense. You could say by linear algebra, but linear algebra's got a little more to it than this.

So now I know M, and now I know N. So now I know the answer. y is M, so M is minus aN. Oh, well, I have to figure out what N is, here.

What is N? This is giving me N, but I better figure it out. What is N from that first equation? And then I'll plug in. And then I'm quit.

AUDIENCE: [INAUDIBLE].

PROFESSOR: 1 over, yeah.

AUDIENCE: 1 plus a squared.

PROFESSOR: 1 plus a squared, good. Because that term goes over there, and we have 1 plus a squared. So now y is M cosine t. So M is minus aN. So minus aN is 1 over 1 plus a squared cosine t. Is that right? That was the cosines.

And we had N sine t. But N is just 1-- I think I just add the sine t. Have I got it? I think so. Here is the N sine t, and here is the M cos t. It was just algebra.

Typical of these problems, there's a little thinking and then some algebra. The thinking led us to this. The thinking led us to the fact we needed cosines in there, as well as cosines. But then once we did it, then the thinking said, OK, separately match the cosine terms and the sine term. And then do the algebra.

Now, I just want to do this with complex. So y prime equals ay plus e to the it. To get an idea, you see the two. And then I have to talk about it. You see, I'm only going to go part way with this and then save it for Wednesday.

But if I see this, what solution do I assume? This is like an e to the st. I assume y is some Y e to the it. See, I don't have cosines and sines anymore. I have e to the it. And if I take the derivative of e to the it, I'm still in the e to the it world.

So I do this. I plug it in. Uh-huh, let me leave that for Wednesday. We have to have some excitement for Wednesday. So we'll get a complex answer, and then we'll take the real part to solve that problem.

So we've got two steps, one way or the other way. Here, we had two steps because we had to let sines sneak in. Here, we have two steps because I could solve it, and you could solve that right away. But then you have to take the real part. I'll leave that.

Is there questions? Do you want me to recap quickly what we've done.

AUDIENCE: Yes.

PROFESSOR: I try to leave on the board enough to make a recap possible. Everything was about that equation. We have only solved-- I shouldn't say only-- we have solved the constant coefficient, model constant coefficient, first order equation. Wednesday comes nonlinear equation. This one today was strictly linear.

So what did we do? We solved this equation, first of all, for q equal 1; secondly, for q equal e to the st; thirdly, for q equal a step; fourthly for q equal-- where is it? Where is that delta of t? Maybe it's here. Ah, it got erased. So the fourth guy was y prime equal ay plus delta of t, or delta of t minus capital T. So those were our four examples.

And then what did we finally do? So if we're recapping, compressing, we're compressing everything into two minutes. We solved those four examples, and then we solved the general problem. And when we solved the general problem, that gave us this integral, which my whole goal was that you should understand that this should seem right to you.

This is adding up the value at time t from all the inputs at different times s. So to add them up, we integrate from 0 to t. And finally, we returned to the question of cos t, all important question. But awkward question, because we needed to let sine t in there too.