Differential equation and solution
aka Pure and Warping Torsion
aka Free and Restrained Warping

ref: Hughes 6.1 (eqn 6.1.18)

the development of warping torsion up to this point was assumed to be "pure" or "free" i.e. it was the only effect on a beam and it's behavior was unrestrained. this led us to state ϕ'' and ϕ''' were constant. the development of St. Venant's torsion in 13.10 was developed the same way, ϕ' constant. we will now address the situation where boundary conditions may affect one or both of these effects.

combined torsional resistance determined by:

$$M_x = M_x^{St\_V} + M_x^{w}$$

$M_x^{St\_V}$ is St. Venant's torsion = $G \cdot K_T \cdot \phi'$

M_x^{w} is warping torsion = $-E \cdot I_{\omega\omega} \cdot \phi'''$

M_x is internal or external concentrated torque T_ω

$$M_x = G \cdot K_T \cdot \phi' - E \cdot I_{\omega\omega} \cdot \phi''' \qquad (1)$$

uniform (distributed) torque m_x (torque per unit length) is related to Mx

equilibrium element => $-m_x = \frac{d}{dx} M_x$ =>

differentiating (1) =>

$$m_x = E \cdot I_{\omega\omega} \cdot \phi'''' - G \cdot K_T \cdot \phi'' \qquad (2)$$

solution of (1) has homogeneous and particular solution. rewriting:

$$\phi''' - \frac{G \cdot K_T}{E \cdot I\omega\omega} \cdot \phi' = \frac{-M_x}{E \cdot I_{\omega\omega}} \qquad \text{let } \lambda^2 = \frac{G \cdot K_T}{E \cdot I_{\omega\omega}}$$

homogeneous => $\phi''' - \lambda^2 \cdot \phi' = 0$ assume solution $\phi_H := e^{m \cdot x}$

$$\frac{d^3}{dx^3}\phi_H - \lambda^2 \cdot \frac{d}{dx}\phi_H \rightarrow m^3 \cdot \exp(m \cdot x) - \lambda^2 \cdot m \cdot \exp(m \cdot x) \Rightarrow m^3 - \lambda^2 \cdot m = m \cdot \left(m^2 - \lambda^2\right) = 0$$

roots $m := 0$ $m := \lambda$ $m := -\lambda$

homogeneous solution => $\phi_H := c_1 \cdot e^0 + c_2 \cdot e^{\lambda \cdot x} + c_2 \cdot e^{-\lambda \cdot x}$

particular solution assume $\phi_P := A \cdot x$ $\phi''' - \frac{G \cdot K_T}{E \cdot I\omega\omega} \cdot \phi' = \frac{-M_x}{E \cdot I_{\omega\omega}}$

$$\frac{d^3}{dx^3}\phi_P - \lambda^2 \cdot \frac{d}{dx}\phi_P \rightarrow -\lambda^2 \cdot A$$ is a solution <=> $A := \frac{M_x}{\lambda^2 \cdot E \cdot I_{\omega\omega}}$ $-\lambda^2 \cdot A \rightarrow \frac{-M_x}{E \cdot I_{\omega\omega}}$

therefore:

$$\phi(x) := c_1 + c_2 \cdot e^{\lambda \cdot x} + c_2 \cdot e^{-\lambda \cdot x} + \frac{M_x}{\lambda^2 \cdot E \cdot I_{\omega\omega}} \cdot x$$ which can be rewritten as

$$\phi(x) := A + B \cdot \cosh(\lambda \cdot x) + C \cdot \sinh(\lambda \cdot x) + \frac{M_x}{\lambda^2 \cdot E \cdot I_{\omega\omega}} \cdot x$$

similarly equation (2) => $\phi^{IV} - \lambda^2 \cdot \phi'' = \frac{m_x}{E \cdot I_{\omega\omega}}$

homogeneous => $\phi^{IV} - \lambda^2 \cdot \phi'' = 0$ assume solution $\phi_H := e^{m2 \cdot x}$

$$\frac{d^4}{dx^4}\phi_H - \lambda^2 \cdot \frac{d^2}{dx^2}\phi_H \rightarrow m2^4 \cdot \exp(m2 \cdot x) - \lambda^2 \cdot m2^2 \cdot \exp(m2 \cdot x)$$ => $m2^4 - \lambda^2 \cdot m2^2 = 0$

roots $m2 := 0$ $m2 := 0$ $m2 := \lambda$ $m2 := -\lambda$ (double root)

homogeneous solution => $\phi_H := c_1 \cdot e^0 + c_2 \cdot x \cdot e^0 + c_3 \cdot e^{\lambda \cdot x} + c_4 \cdot e^{-\lambda \cdot x}$

particular solution assume $\phi_P := A1 \cdot x^2 + B \cdot x^3$

$$\frac{d^4}{dx^4}\phi_P - \lambda^2 \cdot \frac{d^2}{dx^2}\phi_P \rightarrow -\lambda^2 \cdot (2 \cdot A1 + 6 \cdot B \cdot x)$$

$$-\lambda^2 \cdot (2 \cdot A1 + 6 \cdot B \cdot x) = \frac{m_x}{E \cdot I_{\omega\omega}}$$ is a solution <=> $B := 0$ and $A1 := \frac{-m_x}{2 \cdot \lambda^2 \cdot E \cdot I_{\omega\omega}}$

$$\phi_P := \frac{-m_x}{2\cdot\lambda^2\cdot E\cdot I_{\omega\omega}}\cdot x^2 \qquad \frac{d^4}{dx^4}\phi_P - \lambda^2\cdot\frac{d^2}{dx^2}\phi_P \rightarrow \frac{m_x}{E\cdot I_{\omega\omega}} \qquad \text{check}$$

therefore:

$$\phi_H := c_1 + c_2\cdot x + c_3\cdot e^{\lambda\cdot x} + c_4\cdot e^{-\lambda\cdot x} - \frac{m_x}{2\cdot\lambda^2\cdot E\cdot I_{\omega\omega}}\cdot x^2$$

which can be rewritten as

$$\phi(x) := A + B\cdot x + C\cdot\cosh(\lambda\cdot x) + D\cdot\sinh(\lambda\cdot x) - \frac{m_x}{2\cdot\lambda^2\cdot E\cdot I_{\omega\omega}}\cdot x^2$$

boundary conditions for various situations:

fixed end	$\phi = 0$	no twist	$\phi' = 0$	no slope	
pinned end	$\phi = 0$	no twist	$Bi = 0$	free warping	
free end	$Bi = 0$	free warping	$\phi''' = 0$	no warping shear	
continuous supports	$\phi = 0$	no twist	$\phi_{l'} = \phi_{r'}$	$Bi_l = Bi_r$	continuous
transition point within span	$\phi_l = \phi_r$		$\phi_{l'} = \phi_{r'}$	$Bi_l = Bi_r$	continuous
general	$\phi'' = 0$	free end from bending			

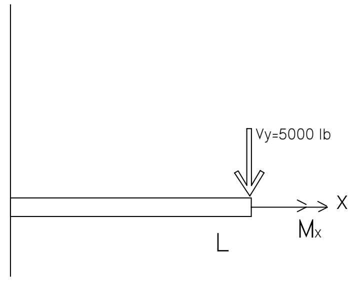
for a visual of Bi the bimoment see figure 6.13 in text

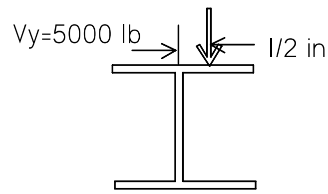
Problem - Torsional response of Cantilever Girder

general solution for end moment Mx and fixed at x = 0 (cantilever)

$x = 0 \qquad \phi = \phi' = 0$

$x = L \qquad \phi'' := 0$





from restrained_torsion.mcd $\quad \phi(x) := A + B\cdot\cosh(\lambda\cdot x) + C\cdot\sinh(\lambda\cdot x) + \dfrac{M_x}{\lambda^2\cdot E\cdot I_{\omega\omega}}\cdot x$

$$\phi(0) \rightarrow \frac{M_x}{\lambda^2\cdot E\cdot I_{\omega\omega}} \qquad A + B = 0$$

$$\phi\_pr(x) := \frac{d}{dx}\phi(x) \qquad \phi\_pr(x) \rightarrow C\cdot\cosh(\lambda\cdot x)\cdot\lambda + \frac{M_x}{\lambda^2\cdot E\cdot I_{\omega\omega}}$$

$\phi' = 0$

$$\phi\_pr(0) \left| \begin{array}{l} \text{simplify} \\ \text{collect}, C \end{array} \right. \rightarrow C\cdot\lambda + \frac{M_x}{\lambda^2\cdot E\cdot I_{\omega\omega}} \qquad C\cdot\lambda + \frac{M_x}{\lambda^2\cdot E\cdot I_{\omega\omega}} = 0$$

or ... substituting $\lambda^2 = \dfrac{G\cdot K_T}{E\cdot I_{\omega\omega}}$ $\qquad C\cdot\lambda + \dfrac{M_x}{G\cdot K_T} = 0$

free end => $\phi'' = 0$

$$\phi\_db\_pr(x) := \frac{d^2}{dx^2}\phi(x) \qquad \phi\_db\_pr(x) \rightarrow C\cdot\sinh(\lambda\cdot x)\cdot\lambda^2$$

$$\phi\_db\_pr(L) \rightarrow C\cdot\sinh(\lambda\cdot L)\cdot\lambda^2 \qquad B\cdot\cosh(\lambda\cdot L)\cdot\lambda^2 + C\cdot\sinh(\lambda\cdot L)\cdot\lambda^2 = 0$$

or ..... $\quad B + C\cdot\tanh(\lambda\cdot L) = 0$

Given $\quad A + B = 0 \qquad C\cdot\lambda + \dfrac{M_x}{G\cdot K_T} = 0 \qquad B + C\cdot\tanh(\lambda\cdot L) = 0$

$$\text{Find}(A,B,C) \rightarrow \begin{pmatrix} \dfrac{-M_x}{\lambda\cdot G\cdot K_T}\cdot\tanh(\lambda\cdot L) \\ \dfrac{M_x}{\lambda\cdot G\cdot K_T}\cdot\tanh(\lambda\cdot L) \\ \dfrac{-M_x}{\lambda\cdot G\cdot K_T} \end{pmatrix}$$

$$B := \frac{M_x}{\lambda \cdot G \cdot K_T} \cdot \tanh(\lambda \cdot L) \qquad A := -B \qquad C := \frac{-M_x}{\lambda \cdot G \cdot K_T}$$

$$\phi(x) := A + B \cdot \cosh(\lambda \cdot x) + C \cdot \sinh(\lambda \cdot x) + \frac{M_x}{\lambda^2 \cdot E \cdot I_{\omega\omega}} \cdot x$$

$$\phi(x) \rightarrow \frac{-M_x}{\lambda \cdot G \cdot K_T} \cdot \tanh(\lambda \cdot L) + \frac{M_x}{\lambda \cdot G \cdot K_T} \cdot \tanh(\lambda \cdot L) \cdot \cosh(\lambda \cdot x) - \frac{M_x}{\lambda \cdot G \cdot K_T} \cdot \sinh(\lambda \cdot x) + \frac{M_x}{\lambda^2 \cdot E \cdot I_{\omega\omega}} \cdot x$$

substituting for A, B, C, and λ^2

$$\phi(x) := \frac{M_x}{\lambda \cdot G \cdot K_T} \cdot \tanh(\lambda \cdot L) \cdot (\cosh(\lambda \cdot x) - 1) - \frac{M_x}{\lambda \cdot G \cdot K_T} \cdot \sinh(\lambda \cdot x) + \frac{M_x}{G \cdot K_T} \cdot x$$

$$\phi(x) := \frac{M_x}{\lambda \cdot G \cdot K_T} \cdot [\tanh(\lambda \cdot L) \cdot (\cosh(\lambda \cdot x) - 1) - \sinh(\lambda \cdot x) + \lambda \cdot x]$$

$$\frac{d}{dx}\phi(x) \text{ collect}, \lambda \rightarrow \frac{M_x}{G \cdot K_T} \cdot (\tanh(\lambda \cdot L) \cdot \sinh(\lambda \cdot x) - \cosh(\lambda \cdot x) + 1)$$

$$\phi\_pr(x) := \frac{M_x}{G \cdot K_T} \cdot (\tanh(\lambda \cdot L) \cdot \sinh(\lambda \cdot x) - \cosh(\lambda \cdot x) + 1)$$

$$\frac{d^2}{dx^2}\phi(x) \text{ collect}, \lambda \rightarrow \frac{M_x}{G \cdot K_T} \cdot (\tanh(\lambda \cdot L) \cdot \cosh(\lambda \cdot x) - \sinh(\lambda \cdot x)) \cdot \lambda$$

$$\phi\_db\_pr(x) := \frac{M_x \cdot \lambda}{G \cdot K_T} \cdot (\tanh(\lambda \cdot L) \cdot \cosh(\lambda \cdot x) - \sinh(\lambda \cdot x))$$

$$\frac{d^3}{dx^3}\phi(x) \text{ collect}, \lambda \rightarrow \frac{M_x}{G \cdot K_T} \cdot (\tanh(\lambda \cdot L) \cdot \sinh(\lambda \cdot x) - \cosh(\lambda \cdot x)) \cdot \lambda^2$$

$$\phi\_tr\_pr(x) := \frac{M_x \cdot \lambda^2}{G \cdot K_T} \cdot (\tanh(\lambda \cdot L) \cdot \sinh(\lambda \cdot x) - \cosh(\lambda \cdot x))$$

let's look at these in general

$$\lambda^2 = \frac{G \cdot K_T}{E \cdot I_{\omega\omega}}$$

$2 < \lambda \cdot L < 5$ $\quad$ $F0(x), F1(x), F2(x), F3(x)$

$L := 1 \quad \lambda := 5 \quad \lambda \cdot L = 5 \quad x := 0, 0.1 .. L$

above equations factoring out

$F0(x) := \tanh(\lambda \cdot L) \cdot (\cosh(\lambda \cdot x) - 1) - \sinh(\lambda \cdot x) + \lambda \cdot x$ $\quad$ $\frac{M_x}{G \cdot K_T \cdot \lambda}$, $\phi(x)$ ~ total twist

$F1(x) := \tanh(\lambda \cdot L) \cdot \sinh(\lambda \cdot x) - \cosh(\lambda \cdot x) + 1$ $\quad$ $\frac{M_x}{G \cdot K_T}$, $\phi'(x)$ ~ St V torsion

$F2(x) := \tanh(\lambda \cdot L) \cdot \cosh(\lambda \cdot x) - \sinh(\lambda \cdot x)$ $\quad$ $\frac{M_x \cdot \lambda}{G \cdot K_T}$, $\phi''(x)$ ~ axial stress warping torsion

$F3(x) := \tanh(\lambda \cdot L) \cdot \sinh(\lambda \cdot x) - \cosh(\lambda \cdot x)$ $\quad$ $\frac{M_x \cdot \lambda^2}{G \cdot K_T}$, $\phi'''(x)$ ~ shear stress warping torsion

respectively