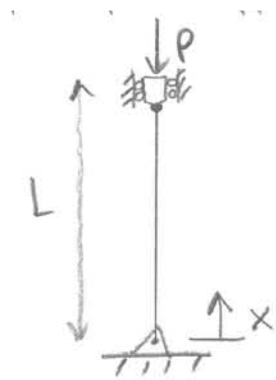
Recitation 7: Column Buckling Solutions Using Equilibrium

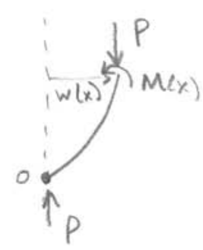
Example 1

Find P_c for a pin-pin supported column.



Global Equilibrium:

$$\overset{\curvearrowleft}{+}\Sigma M_0 = M(x) - Pw(x) = 0$$
$$M(x) = Pw(x)$$



Constitutive Law for Beams/Columns: $M = EI\kappa = -EIw''$

$$\Rightarrow -EIw'' = Pw \tag{7.1}$$

$$\text{or } \boxed{w'' + \frac{P}{EI}w = 0} \quad \text{Governing D.E.} \tag{7.2}$$

Recall Diff Eqs

Characteristic Eqn: $\lambda^2 + \dfrac{P}{EI} = 0 \rightarrow \lambda = \pm i\sqrt{P/EI}$

General solution: $w = C_1 \sin\sqrt{\dfrac{P}{EI}}x + C_2 \cos\sqrt{\dfrac{P}{EI}}x$

Apply Boundary Conditions:
$w(0) = 0 : C_1(0) + C_2(1) = 0 \rightarrow C_2 = 0$

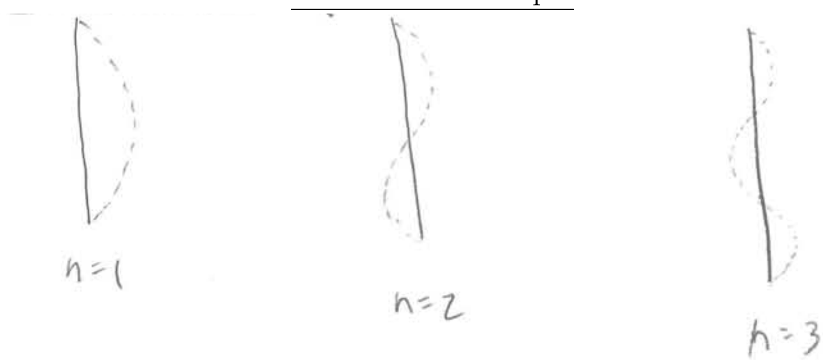
$$w(L) = 0 : C_1 \sin\sqrt{\frac{P}{EI}}L = 0 \rightarrow \text{either } C_1 = 0 \text{ (trivial soln.)}$$
$$\text{or } \sqrt{\frac{P}{EI}}L = n\pi$$

$$\Rightarrow \frac{P}{EI}L^2 = n^2\pi^2 \rightarrow P = \frac{n^2\pi^2 EI}{L^2}$$

$$P \text{ is minimum when } n = 1 \Rightarrow \boxed{P_C = \frac{\pi^2 EI}{L^2}}$$

Note: Buckling solutions do *NOT* give us the deflection amplitude (C_1). The calculated P_C could result in *any* amplitude.

Different mode shapes



Example 2:

Find P_C for a clamped-clamped column.

Different mode shapes

Local equilibrium: $EI\frac{d^4w}{dx^4} + P\frac{d^2w}{dx^2} = 0$

Characteristic eqn: $\lambda^4 + \frac{P}{EI}\lambda^2 = 0 \rightarrow \lambda_1 = 0$

$$\lambda_2 = 0$$

$$\lambda_3 = i\sqrt{\frac{P}{EI}}$$

$$\lambda_3 = -i\sqrt{\frac{P}{EI}}$$

General solution: $w(x) = C_1 \sin\sqrt{\frac{P}{EI}}x + C_2 \cos\sqrt{\frac{P}{EI}}x + C_3x + C_4$

then $w'(x) = C_1\sqrt{\frac{P}{EI}} \cos\sqrt{\frac{P}{EI}}x - C_2\sqrt{\frac{P}{EI}} \sin\sqrt{\frac{P}{EI}}x + C_3$

Apply BSs:

① $w(0) = 0$: $C_1(0) + C_2(1) + C_3(0) + C_4 = 0 \rightarrow C_2 = -C_4$

② $w'(0) = 0$: $C_1\sqrt{\frac{P}{EI}}(1) - C_2\sqrt{\frac{P}{EI}}(0) + C_3 = 0 \rightarrow C_3 = -C_1\sqrt{\frac{P}{EI}}$

③ $w(L) = 0$: $C_1 \sin\sqrt{\frac{P}{EI}}L + C_2 \cos\sqrt{\frac{P}{EI}}L - C_1\sqrt{\frac{P}{EI}}L - C_2 = 0$

④ $w'(L) = 0$: $C_1\sqrt{\frac{P}{EI}} \cos\sqrt{\frac{P}{EI}}L - C_2\sqrt{\frac{P}{EI}} \sin\sqrt{\frac{P}{EI}}L - C_1\sqrt{\frac{P}{EI}} = 0$

Matrix form of eons ③ and ④:

$$\nearrow \omega$$

$$\begin{bmatrix} \sin\sqrt{\frac{P}{EI}}L - \sqrt{\frac{P}{EI}}L & \cos\sqrt{\frac{P}{EI}}L - 1 \\ \sqrt{\frac{P}{EI}}(\cos\sqrt{\frac{P}{EI}}L - 1) & -\sqrt{\frac{P}{EI}}\sin\sqrt{\frac{P}{EI}}L \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7.3}$$

Non-trivial solution only if $\det[\dots] = 0$

$$(\sin\omega L - \omega L)(-\omega\sin\omega L) - \omega(\cos\omega L - 1)(\cos\omega L - 1) = 0 \tag{7.4a}$$

$$-\omega\sin^2\omega L + \omega^2 L\sin\omega L - \omega\cos^2\omega L + 2\omega\cos\omega L - \omega = 0 \tag{7.4b}$$

$$-2\omega + \omega^2 L\sin\omega L + 2\omega\cos\omega L = 0 \tag{7.4c}$$

$$\omega(-2 + \omega L\sin\omega L + 2\cos\omega L) = 0 \tag{7.4d}$$

$$\Rightarrow \quad \omega = 0 \quad \text{or} \quad (-2 + \omega L\sin\omega L + 2\cos\omega L) = 0 \tag{7.5}$$

$$\boxed{\begin{aligned} \text{Recall: } \sin 2\theta &= 2\sin\theta\cos\theta \\ \cos 2\theta &= 1 - 2\sin^2\theta \end{aligned}} \quad \text{Let } \theta = \frac{\omega L}{2} \tag{7.6}$$

Then

$$-2 + \omega L 2\sin\left(\frac{\omega L}{2}\right)\cos\left(\frac{\omega L}{2}\right) + 2\left[1 - 2\sin^2\left(\frac{\omega L}{2}\right)\right] = 0 \tag{7.7}$$

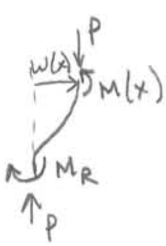
$$\sin\left(\frac{\omega L}{2}\right)\left[2\omega L\cos\left(\frac{\omega L}{2}\right) - 4\sin\left(\frac{\omega L}{2}\right)\right] = 0 \tag{7.8}$$

$$\sin\left(\frac{\omega L}{2}\right) = 0 \;\rightarrow\; \frac{\omega L}{2} = n\pi \;\rightarrow\; \sqrt{\frac{\omega L}{2}} = \frac{2n\pi}{L} \tag{7.9}$$

$$\boxed{P_C = \frac{4n^2\pi^2 EI}{L^2}} \tag{7.10}$$

Alternative Method

Glabal equilibrium, with unknown M_R:



$$\overset{\curvearrowleft}{+}\Sigma M_0 : \quad -M_R + M(x) - Pw(x) = 0 \tag{7.11}$$

$$M(x) = M_R + Pw(x) = -EIw'' \tag{7.12}$$

So $\boxed{w'' + \frac{P}{EI}w = -\frac{M_R}{EI}}$ Inhomogeneous O.D.E.

Solution: $w = w_{\mathrm{h}} + w_{\mathrm{p}}$

$\uparrow$ homogeneous $\nwarrow$ particular

$$w_{\mathrm{h}} = C_1 \sin\sqrt{\frac{P}{EI}}x + C_2 \cos\sqrt{\frac{P}{EI}}x \text{ (as before in example 1)}$$

$$w_{\mathrm{p}} = C_3, \text{ where } C_3 = -\frac{M_{\mathrm{R}}}{P} \leftarrow \text{(Not necessarily constant - may be a function of } \omega\text{)}$$

$$\left(\sqrt{\tfrac{P}{EI}} = \omega\right)$$

Then $w(x) = C_1 \sin\sqrt{\frac{P}{EI}}x + C_2 \cos\sqrt{\frac{P}{EI}}x - \frac{M_{\mathrm{R}}}{P}$

$$w' = C_1\omega \cos\omega x - C_2\omega \sin\omega x$$

Apply BCs:

$w(0) = 0$: $C_2 = \frac{M_{\mathrm{R}}}{P}$

$w'(0) = 0$: $C_1 w = 0 \rightarrow C_1 = 0$

$w(L) = 0$: $\frac{M_{\mathrm{R}}}{P}\cos\omega L = \frac{M_{\mathrm{R}}}{P} \rightarrow \cos\omega L = 1 \rightarrow \omega L = 2n\pi$

$$\boxed{\omega = \frac{2n\pi}{L}} \quad \text{As Before}$$