Ocean Acoustic Theory

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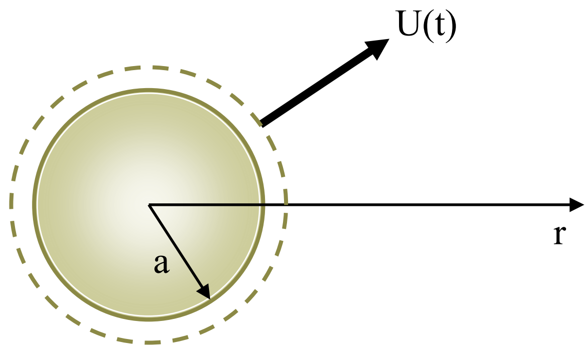


Vibrating sphere in an infinite fluid medium.

Source in Unbounded Medium

Frequency Domain

$$u_r(a) = U(\omega)\,.$$

Spherical geometry solution

$$\begin{aligned}\psi(r) &= A\,\frac{e^{ikr}}{r}\,,\\ u_r(r) &= \frac{\partial\psi(r)}{\partial r} = A\,e^{ikr}\left(\frac{ik}{r} - \frac{1}{r^2}\right).\end{aligned}$$

Simple point source: $ka \ll 1$

$$\begin{aligned}u_r(a) &= A\,e^{ika}\,\frac{ika-1}{a^2} \simeq -\frac{A}{a^2}\,,\\ A &= -a^2\,U(\omega)\,.\\ &\Rightarrow\\ \psi(r) &= -S_\omega\,\frac{e^{ikr}}{4\pi r}\,.\\ S_\omega &= 4\pi a^2\,U(\omega)\end{aligned}$$

Green's function

,

$$g_\omega(r,0) = \frac{e^{ikr}}{4\pi r} \,,$$

Source at r_0

$$g_\omega(\mathbf{r},\mathbf{r}_0) = \frac{e^{ikR}}{4\pi R} \,, \quad R = |\mathbf{r}-\mathbf{r}_0| \,.$$

Helmholtz Equation for Green's function

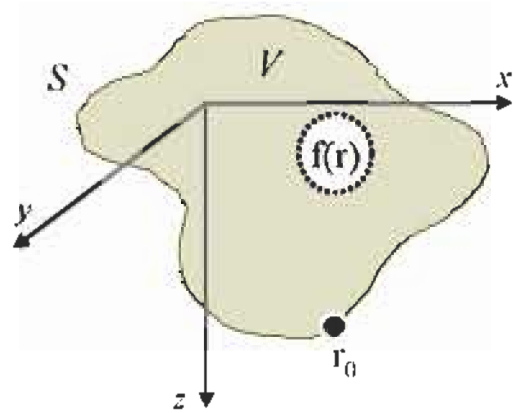
$$\left[\nabla^2 + k^2\right] g_\omega(\mathbf{r},\mathbf{r}_0) = -\delta(\mathbf{r}-\mathbf{r}_0) \,,$$

Integrate over spherical volume V of radius $\epsilon \to 0$:

$$\begin{aligned}
\int_V -\delta(\mathbf{r}-\mathbf{r}_0)dV &= -1 \\
\int_V k^2 g_\omega(\mathbf{r},\mathbf{r}_0)dV &\to_{\epsilon\to 0} 0 \\
\int_V \nabla^2 g_\omega(\mathbf{r},\mathbf{r}_0)dV &= \int_S \frac{\partial}{\partial R} g_\omega(\mathbf{r},\mathbf{r}_0)dS \\
&= \int_S \left[-\frac{e^{ik\epsilon}}{4\pi\epsilon^2} + \frac{ike^{ik\epsilon}}{4\pi\epsilon}\right] dS \\
&= 4\pi\epsilon^2 \left[-\frac{e^{ik\epsilon}}{4\pi\epsilon^2} + \frac{ike^{ik\epsilon}}{4\pi\epsilon}\right] \to_{\epsilon\to 0} -1
\end{aligned}$$

Reciprocity

$$g_\omega(\mathbf{r},\mathbf{r}_0) = g_\omega(\mathbf{r}_0,\mathbf{r}) \,,$$



Sources in a volume V bounded by the surface S.

Source in Bounded Medium

Inhomogeneous Helmholtz Equation

$$\left[\nabla^2 + k^2\right] \psi(\mathbf{r}) = f(\mathbf{r}) \,.$$

General Green's Function

$$G_\omega(\mathbf{r}, \mathbf{r}_0) = g_\omega(\mathbf{r}, \mathbf{r}_0) + H_\omega(\mathbf{r}) \,,$$

$$\left[\nabla^2 + k^2\right] H_\omega(\mathbf{r}) = 0 \,.$$

$$\Rightarrow$$

$$\left[\nabla^2 + k^2\right] G_\omega(\mathbf{r}, \mathbf{r}_0) \;=\; -\delta(\mathbf{r} - \mathbf{r}_0) \,.$$

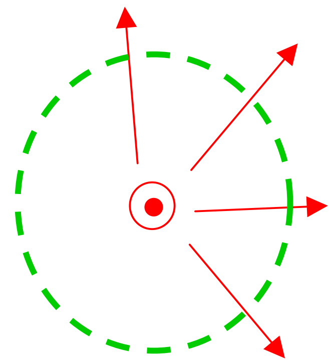
Green's Theorem

$$\psi(\mathbf{r}) = \int_S \left[G_\omega(\mathbf{r}, \mathbf{r}_0) \frac{\partial \psi(\mathbf{r}_0)}{\partial \mathbf{n}_0} - \psi(\mathbf{r}_0) \frac{\partial G_\omega(\mathbf{r}, \mathbf{r}_0)}{\partial \mathbf{n}_0} \right] dS_0 - \int_V f(\mathbf{r}_0) \, G_\omega(\mathbf{r}, \mathbf{r}_0) \, dV_0,$$

Source in infinite medium

$$\psi(\mathbf{r}) = -\int_V f(\mathbf{r}_0)\, g_\omega(\mathbf{r},\mathbf{r}_0)\, dV_0\,.$$

For any imaginary surface enclosing the sources:



$$\int_S \left[g_\omega(\mathbf{r},\mathbf{r}_0)\,\frac{\partial \psi(\mathbf{r}_0)}{\partial \mathbf{n}_0} - \psi(\mathbf{r}_0)\,\frac{\partial g_\omega(\mathbf{r},\mathbf{r}_0)}{\partial \mathbf{n}_0} \right] dS_0 = 0\,.$$

$$\Rightarrow \quad \int_S \frac{e^{ikR}}{4\pi R}\left[\frac{\partial \psi(\mathbf{r}_0)}{\partial R} - ik\,\psi(\mathbf{r}_0)\right] dS_0 = 0\,.$$

Radiation condition

$$R\left[\frac{\partial}{\partial R} - ik\right]\psi(\mathbf{r}_0) \to 0\,, \quad R = |\mathbf{r}-\mathbf{r}_0| \to \infty\,.$$

$$\Rightarrow \quad \int_S \frac{e^{ikR}}{4\pi R}\left[\frac{\partial \psi(\mathbf{r}_0)}{\partial R} - ik\,\psi(\mathbf{r}_0)\right] dS_0 = 0\,.$$

Radiation condition

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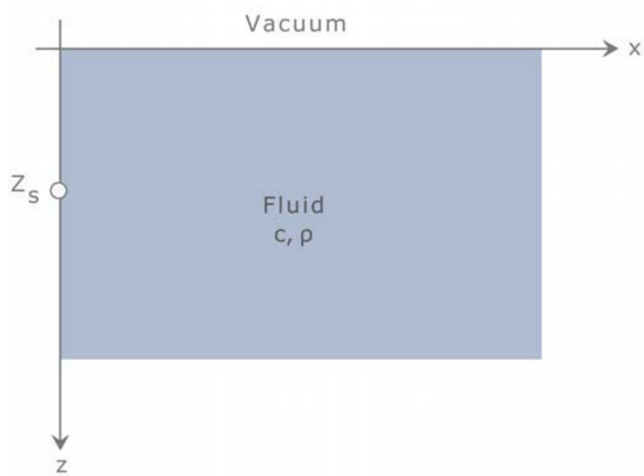


Point Source in Fluid Halfspace

Acoustic Pressure

$$p(\mathbf{r}) = \rho\omega^2\,\psi(\mathbf{r})\,,$$

Pressure-release boundary condition

$$\psi(\mathbf{r}_0) \equiv 0\,, \quad \mathbf{r}_0 = (x, y, 0)\,.$$

Green's theorem

$$\psi(\mathbf{r}) = \int_S G_\omega(\mathbf{r},\mathbf{r}_0)\,\frac{\partial\psi(\mathbf{r}_0)}{\partial\mathbf{n}_0}\,dS_0 - \int_V f(\mathbf{r}_0)\,G_\omega(\mathbf{r},\mathbf{r}_0)\,dV_0\,.$$

Simple point source

$$f(\mathbf{r}_0) = S_\omega\,\delta(\mathbf{r}_0 - \mathbf{r}_s)\,.$$

Green's Function

$$\begin{aligned}
\text{Choose}\quad G_\omega(\mathbf{r},\mathbf{r}_0) &\equiv 0 \quad \text{for} \quad \mathbf{r}_0 = (x,y,0) \\
G_\omega(\mathbf{r},\mathbf{r}_0) &= g_\omega(\mathbf{r},\mathbf{r}_0) + H_\omega(\mathbf{r}) \\
&= \frac{e^{ikR}}{4\pi R} - \frac{e^{ikR'}}{4\pi R'} \\
&\Rightarrow \\
\psi(\mathbf{r}) &= -S_\omega\,G_\omega(\mathbf{r},\mathbf{r}_s)\,.
\end{aligned}$$

with

$$\begin{aligned}
R &= \sqrt{(x-x_s)^2 + (y-y_s)^2 + (z-z_s)^2}\,, \\
R' &= \sqrt{(x-x_s)^2 + (y-y_s)^2 + (z+z_s)^2}\,.
\end{aligned}$$

Acoustic Pressure

$$p(\mathbf{r}) = \rho\omega^2\psi(\mathbf{r}) = -\rho\omega^2 S_\omega\left[\frac{e^{ikR}}{4\pi R} - \frac{e^{ikR'}}{4\pi R'}\right]\,,$$

Transmission Loss

$$\mathrm{TL}(\mathbf{r},\mathbf{r}_s) = -20\log_{10}\left|\frac{p(\mathbf{r},\mathbf{r}_s)}{p(R=1m)}\right| \;,$$

$$\begin{aligned} p(R=1) &= \rho\omega^2\,\psi(\omega,R=1) \\ &= -\rho\omega^2 S_\omega \frac{e^{ik}}{4\pi} = 1 \\ &\Rightarrow \\ S_\omega &= -\frac{4\pi}{\rho\omega^2} \end{aligned}$$

Transmission Loss Pressure

$$P(\mathbf{r},\mathbf{r}_s) = \frac{p(\mathbf{r},\mathbf{r}_s)}{p(R=1m)} \;,$$

where

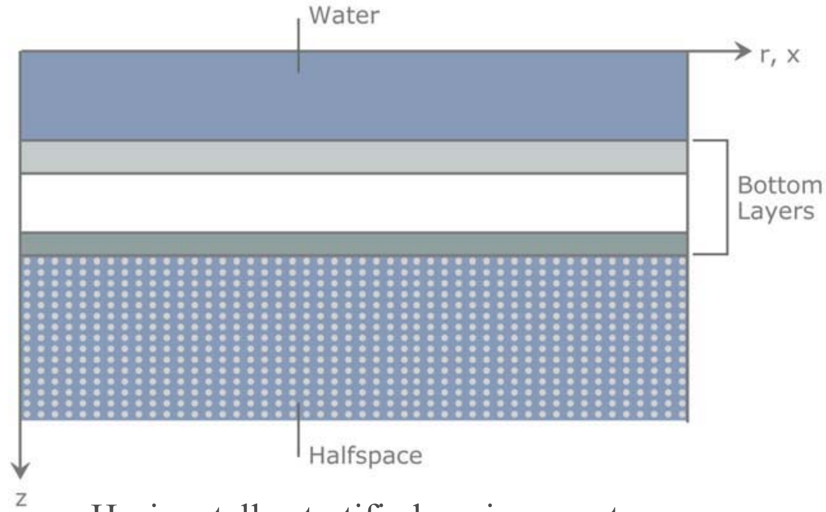
$$\left[\nabla^2 + k^2\right]\Psi(\mathbf{r},\mathbf{r}_s) = -\frac{4\pi}{\rho\omega^2}\,\delta(\mathbf{r}-\mathbf{r}_s) \;.$$

Transmission Loss Helmholtz Equation

$$\left[\nabla^2 + k^2\right]P(\mathbf{r},\mathbf{r}_s) = -4\pi\,\delta(\mathbf{r}-\mathbf{r}_s) \;.$$

Density Variations

$$\rho\nabla\cdot\left[\rho^{-1}\,\nabla P(\mathbf{r},\mathbf{r}_s)\right] + k^2 P(\mathbf{r},\mathbf{r}_s) = -4\pi\,\delta(\mathbf{r}-\mathbf{r}_s) \;.$$



Horizontally stratified environment

Layered Media and Waveguides

Integral Transform Solution

Helmholtz Equation - Layer n

$$\left[\nabla^2 + k_n^2(z)\right] \psi(\mathbf{r}) = f(\mathbf{r}) \, ,$$

Interface Boundary Conditions

$$B\left[\psi(\mathbf{r})\right]|_{z=z_n} = 0 \, , \quad n = 1 \cdots N \, ,$$

Plane problems: Fourier Transform Solution

$$
\begin{aligned}
f(x,z) &= \int_{-\infty}^{\infty} f(k_x,z)\, e^{ik_x x}\, dk_x \;, \\
f(k_x,z) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x,z)\, e^{-ik_x x}\, dx \;,
\end{aligned}
$$

Depth-Separated Wave Equation

$$
\left[\frac{d^2}{dz^2} + (k^2 - k_x^2)\right] \psi(k_x,z) = S_\omega \,\frac{\delta(z-z_s)}{2\pi} \;.
$$

Depth-Separated Green's Function

$$
\psi(k_x,z) = -S_\omega\, G_\omega(k_x,z,z_s) = -S_\omega\left[g_\omega(k_x,z,z_s) + H_\omega(k_x,z)\right]
$$

$$
\begin{aligned}
\left[\frac{d^2}{dz^2} + (k^2 - k_x^2)\right] g_\omega(k_x,z,z_s) &= -\frac{\delta(z-z_s)}{2\pi} \\
\left[\frac{d^2}{dz^2} + (k^2 - k_x^2)\right] H_\omega(k_x,z) &= 0
\end{aligned}
$$

Interface Boundary Conditions

$$
B\left[\psi(k_x,z_n)\right] = 0 \;.
$$

Axisymmetric Propagation Problems: Hankel Transform Solution

$$f(r,z) = \int_0^\infty f(k_r,z)\, J_0(k_r r)\, k_r\, dk_r \,,$$
$$f(k_r,z) = \int_0^\infty f(r,z)\, J_0(k_r r)\, r\, dr \,,$$

Depth-Separated Wave Equation

$$\left[\frac{d^2}{dz^2} + (k^2 - k_r^2)\right] \psi(k_r,z) = S_\omega \, \frac{\delta(z-z_s)}{2\pi} \,.$$

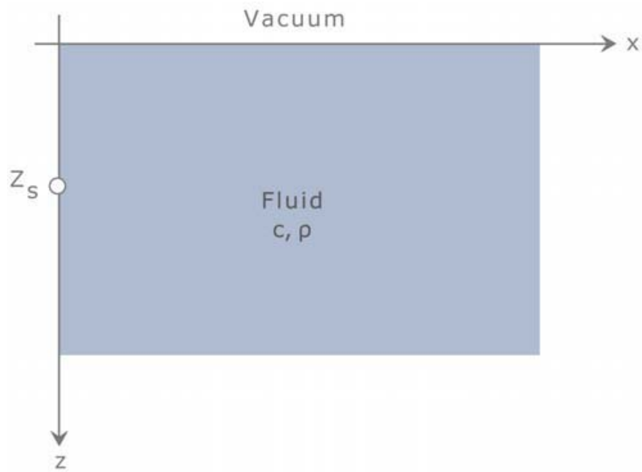
Depth-Separated Green's Function

$$\psi(k_r,z) = -S_\omega\, G_\omega(k_r,z,z_s) = -S_\omega \left[g_\omega(k_r,z,z_s) + H_\omega(k_r,z)\right]$$

$$\left[\frac{d^2}{dz^2} + (k^2 - k_x^2)\right] g_\omega(k_x,z,z_s) = -\frac{\delta(z-z_s)}{2\pi}$$
$$\left[\frac{d^2}{dz^2} + (k^2 - k_x^2)\right] H_\omega(k_x,z) = 0$$

Interface Boundary Conditions

$$B\left[\psi(k_r,z_n)\right] = 0 \,.$$



Point source in a fluid halfspace.

Example: Source in Fluid Halfspace

Homogeneous Solution

$$H_\omega(k_r, z) = A^+(k_r)\, e^{ik_z z} + A^-(k_r)\, e^{-ik_z z}\,,$$

Vertical Wavenumber

$$\begin{aligned} k_z &= \sqrt{k^2 - k_r^2} \\ &= \begin{cases} \sqrt{k^2 - k_r^2}\,, & k_r \le k \\ i\sqrt{k_r^2 - k^2}\,, & k_r > k\,. \end{cases} \end{aligned}$$

Radiation Conditions

$$H_\omega(k_r, z) = \begin{cases} A^+(k_r)\, e^{ik_z z}\,, & z \to +\infty \\ A^-(k_r)\, e^{-ik_z z}\,, & z \to -\infty\,. \end{cases}$$

Source field

$$g_\omega(k_r,z,z_s) = A(k_r)\begin{cases} e^{ik_z(z-z_s)}, & z \ge z_s \\ e^{-ik_z(z-z_s)}, & z \le z_s \end{cases}$$

$$= A(k_r)\, e^{ik_z|z-z_s|} .$$

Integration of depth-separated wave equation over $[z_s-\epsilon, z_s+\epsilon]$:

$$\left[\frac{dg_\omega(k_r,z)}{dz}\right]_{z_s-\epsilon}^{z_s+\epsilon} + O(\epsilon) = -\frac{1}{2\pi} .$$

$$\Rightarrow$$

$$A(k_r) = -\frac{1}{4\pi i k_z}$$

$$\Rightarrow$$

$$g_\omega(k_r,z,z_s) = -\frac{e^{ik_z|z-z_s|}}{4\pi i k_z} .$$

Inverse Hankel Transform - Sommerfeld–Weyl Integral

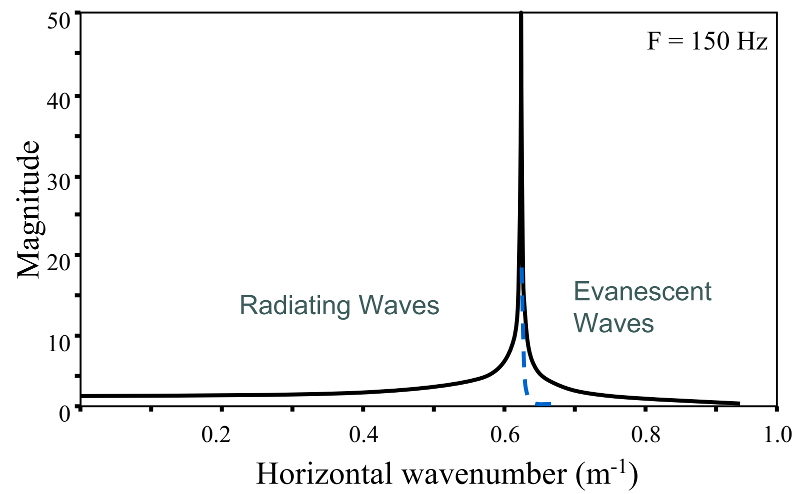
$$g_\omega(r,z,z_s) = \frac{i}{4\pi}\int_0^\infty \frac{e^{ik_z|z-z_s|}}{k_z} J_0(k_r r)\, k_r\, dk_r ,$$

Grazing Angle Representation

$$k_x = k\cos\theta ,$$

$$k_z = k\sin\theta ,$$

$$\frac{dk_x}{d\theta} = -k_z .$$



Magnitude of the depth-dependent Green's function for point source in an infinite medium. Solid curve: $z - z_s = \lambda/10$; dashed curve: $z - z_s = 2\lambda$.

$$\Rightarrow$$

$$g_\omega(\mathbf{r},\mathbf{r}') \simeq \frac{i}{4\pi}\int_{-k}^{k} \frac{e^{ik_z|z-z_s|}}{k_z} e^{ik_x x}\, dk_x$$

$$= \frac{i}{4\pi}\int_0^\pi e^{ik|z-z_s|\sin\theta + ikx\cos\theta}\, d\theta .$$

Halfspace Problem: Surface and Radiation Conditions

$$\psi(k_r, 0) \equiv 0$$
$$\psi(k_r, z) \text{ radiating for } z \rightarrow \infty$$

$$\Rightarrow$$

$$\begin{aligned}\psi(k_r, 0) &= -S_\omega \left[g_\omega(k_r, 0, z_s) + H_\omega(k_r, 0)\right] \\ &= S_\omega \left[\frac{e^{ik_z z_s}}{4\pi i k_z} - A^+(k_r)\right] = 0\,,\end{aligned}$$

Total field

$$\psi(k_r, z) = S_\omega \left[\frac{e^{ik_z|z-z_s|}}{4\pi i k_z} - \frac{e^{ik_z(z+z_s)}}{4\pi i k_z}\right].$$

Loyd-Mirror Minima and Maxima

$$\begin{aligned}\sin\theta_{\max} &= \frac{(2m-1)\,\pi}{2kz_s}\,, \\ \sin\theta_{\min} &= \frac{(m-1)\,\pi}{kz_s}\,.\end{aligned}$$

Free Surface Reflection Coefficient

$$R = -1\,.$$