$$\underline{\underline{M}}\,\ddot{\underline{x}} + (\underline{\underline{G}}+\underline{\underline{C}})\,\dot{\underline{x}} + \underline{\underline{K}}\,\underline{x} = \underline{0}$$

$$\underline{x} = \underline{q} = \dot{\underline{q}} =$$

Damped Oscillations (small) (about $q=q_0$; $x=q-q_0$)

(Holonomic scleronomic System)

(1) $\underline{\underline{M}}\,\ddot{\underline{x}} + \underline{\underline{C}}\,\dot{\underline{x}} + \underline{\underline{K}}\,\underline{x} = \underline{0}$ $\quad\underline{\underline{M}} = \underline{\underline{M}}^T$ pos. def.

$\underline{\underline{K}} = \underline{\underline{K}}^T \rightarrow$ pos. def. $\rightarrow$ because system is Conservative

"damping nature"

If I multiply from the left by $\dot{x}^T$

$$\Rightarrow \frac{d}{dt}\Big[\underbrace{\tfrac{1}{2}\dot{\underline{x}}^T\underline{\underline{M}}\,\dot{\underline{x}} + \tfrac{1}{2}\underline{x}^T\underline{\underline{K}}\,\underline{x}}_{\text{quadratic terms in total energy}}\Big] = -\dot{\underline{x}}^T\underline{\underline{C}}\,\dot{\underline{x}} \Rightarrow \underline{\underline{C}} \text{ is positive semi definite}$$

$$\underline{\underline{C}} = \underbrace{\tfrac{1}{2}(\underline{\underline{C}}+\underline{\underline{C}}^T)}_{\text{Sym.}} + \underbrace{\tfrac{1}{2}(\underline{\underline{C}}-\underline{\underline{C}}^T)}_{\text{Skew-Sym.}}$$

$$= -\dot{x}^T\left(\tfrac{1}{2}(\underline{\underline{C}}+\underline{\underline{C}}^T)\right)\dot{\underline{x}}$$

only the symmetric part contribute to energy dissipation
if $\underline{\underline{C}}$ is positive definite then energy is decreased as long as the velocity is non-zero

Solution of (1) are of the form $\tilde{\underline{x}}(t) = a e^{\lambda t}$, $\lambda \in \mathbb{C}$

$$= a^{(\mathrm{Re}\,\lambda + i\,\mathrm{Im}\,\lambda)t} = a^{(\mathrm{Re}\,\lambda)t}\left[\cos \mathrm{Im}\lambda t + i \sin \mathrm{Im}\lambda t\right.$$

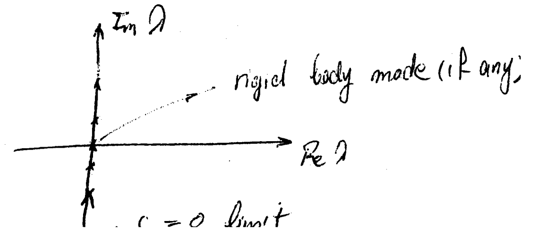
$\Rightarrow$ Substitution into (1):

$$(\lambda^2\underline{\underline{M}} + \lambda\underline{\underline{C}} + \underline{\underline{K}})\,\underline{a} = \underline{0} \qquad (\underline{a}\neq 0)$$

$\rightarrow \det(\lambda^2\underline{\underline{M}} + \lambda\underline{\underline{C}} + \underline{\underline{K}}) = 0$ characteristic equation for λ
with $2n$ roots (real or Complex Conjugate pairs)

Roots on the Complex plane

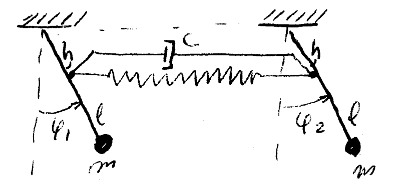
ω_*: undamped natural frequency (for $\underline{\underline{C}} = \underline{\underline{0}}$)



Re $\lambda_i < 0$

Example : Damped Pendulum - Spring System

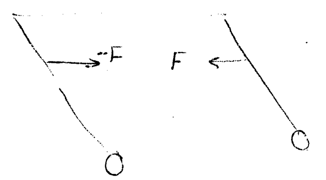
$g \downarrow$



c: damping coeff. for dashpot

we have seen: $\underline{\underline{M}} = \begin{pmatrix} ml^2 & 0 \\ 0 & ml^2 \end{pmatrix}$; $\underline{\underline{K}} = \begin{pmatrix} mgl+Kh^2 & -Kh^2 \\ -Kh^2 & mgl+Kh^2 \end{pmatrix}$

To find $\underline{\underline{C}}$, first identify the generalized (non-potential) force in this system.
active



$$\underline{F} = -c\frac{d}{dt}\left[h(\sin\varphi_2 - \sin\varphi_1)\,\underline{i} - h(\cos\varphi_2 - \cos\varphi_1)\,\underline{j}\right]$$

$$= -ch\left[(\cos\varphi_2\dot\varphi_2 - \cos\varphi_1\dot\varphi_1)\underline{i} - (\sin\varphi_2\dot\varphi_2 + \sin\varphi_1\dot\varphi_1)\underline{j}\right]$$

$\delta W_I = -\underline{F}\cdot\delta\underline{r_1} + \underline{F}\cdot\delta\underline{r_2}$ $\qquad \delta r_1 = \delta(h\sin\varphi\,\underline{i} - h\cos\varphi\,\underline{j})$

$$= h\,\delta\varphi_1(\cos\varphi_1\,\underline{i} + \sin\varphi_1\,\underline{j})$$

$$\delta r_2 = h\,\delta\varphi_2(\cos\varphi_2\,\underline{i} + \sin\varphi_2\,\underline{j})$$

$$\delta W_E = ch^2\delta\varphi_1\left[\cos(\varphi_2-\varphi_1)\dot\varphi_2 - \dot\varphi_1\right] - ch^2\delta\varphi_2\left[\dot\varphi_2 - \cos(\varphi_2-\varphi_1)\dot\varphi_1\right]$$

$\varphi_1 = x_1, \quad \varphi_2 = x_2$

$$\delta W_E = \delta x_1\underbrace{\left[-ch^2(\dot x_1 - \dot x_2) + O(2)\right]}_{Q_{x_1}} + \delta x_2\underbrace{\left[-ch^2(\dot x_2 - \dot x_1) + O(2)\right]}_{Q_{x_2}}$$

For linearized eq. of motion we need $\underbrace{\left.\frac{\partial Q}{\partial \dot\varphi}\right|_{\substack{\varphi=\varphi_0=0\\ \dot\varphi=0}}}_{\left.\frac{\partial Q}{\partial \dot x}\right|_{\substack{x=0\\ \dot x=0}}} = \underbrace{\begin{pmatrix} -ch^2 & ch^2 \\ ch^2 & -ch^2 \end{pmatrix}}_{-\underline{\underline{C}}}$

$$\underline{\underline{C}} = \begin{pmatrix} ch^2 & -ch^2 \\ -ch^2 & ch^2 \end{pmatrix};\ \text{eig values } \lambda_1, \lambda_2 \quad \Leftarrow$$

because it is generalized force corresponding with damping. So $\underline{\underline{C}}$ is the negative of this value

Symmetric

Singular $\quad \det\underline{\underline{C}} = 0 \rightarrow \lambda_1\lambda_2 = 0$

positive semi definite

$$tr(\underline{\underline{C}}) = 2ch^2 = \lambda_1 + \lambda_2$$

$\Rightarrow \lambda_1 = 0,\ \lambda_2 > 0 \rightarrow \underline{\underline{C}}$ pos. semi-definite $\rightarrow$ acceptable

(makes sense physically these are small independent oscillations)

Forced Small Oscillations ($\underset{=}{c} = 0$ for Simplicity)

$$\underset{=}{M}\ddot{\underset{-}{x}} + \underset{=}{k}\underset{-}{x} = \underset{-}{F}(t) = \underset{-}{F}\sin\omega t \quad \text{(Sinusoidal forcing)}$$

pass to modal (principal) Coordinates:

$$\underset{-}{x} = \underset{=}{\Phi}\underset{-}{y}\ ; \quad \underset{=}{\Phi} = [\underset{-}{a}_1, \ldots, \underset{-}{a}_n]$$

As earlier left multiplying (2) by $\underset{=}{\Phi}^T$

$$\underbrace{\underset{=}{\Phi}^T \underset{=}{M}\, \underset{=}{\Phi}}\ddot{\underset{-}{y}} + \underbrace{\underset{=}{\Phi}^T \underset{=}{k}\, \underset{=}{\Phi}}\underset{-}{y} = \underbrace{\underset{=}{\Phi}^T \underset{-}{F}}\sin\omega t$$

$$\begin{pmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_n \end{pmatrix} \quad \begin{pmatrix} k_1 & & 0 \\ & \ddots & \\ 0 & & k_n \end{pmatrix} \quad \underset{-}{f} = \begin{Bmatrix} f_1 \\ \vdots \\ f_n \end{Bmatrix}$$

$$\Rightarrow \ddot{y}_j + \frac{k_j}{m_j} y_j = \frac{f_j}{m_j}\sin\omega t\ ; \qquad j = 1, \ldots, n$$

Solution: $y_j(t) = \overset{hom.}{y_1}(t) + \overset{par.}{y_j}(t)$

$$y_1(t) = C_1 \cos\omega_j t + C_2 \sin\omega_j t\ , \quad \omega_j^2 = \frac{k_j}{m_j}$$

Case (a) $\omega_j \neq \omega \Rightarrow y_j^{par}(t) = A_j \sin\omega t \longrightarrow A_j = \dfrac{f_j/m}{\omega_j^2 - \omega^2}$

$$= \frac{f_j/m}{\omega_j^2 - \omega^2}\sin\omega t$$

Case (b) $\omega_j = \omega$ (resonance) $\Rightarrow y_j^{par}(t) = (A_j + B_j t)\sin\omega t + (C_j + D_j t)\cos\omega t$

$\Rightarrow \cdots$ growing oscillations

Response Diagram