Session 20

Equilibria and their Stability

Assume (1) System is holonomic scleronomic $\Rightarrow L(q,\dot q)$, $q\ (q_1, \dots, q_m)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) - \frac{\partial L}{\partial q} = Q(q,\dot q) \qquad \text{assumed } \frac{\partial Q}{\partial t} = 0$$

2) equilibrium or fixed point : $q = q^\circ = \text{Const}\ (\Rightarrow \dot q = 0)$

$\Rightarrow$ all functions of q & $\dot q$ are Constant in time at equilibria

$$-\frac{\partial L}{\partial q}(q,\dot q) = Q(q^\circ, 0)$$

Stability in Systems with potential forces

$$Q = 0 \qquad -\frac{\partial L}{\partial q}(q^\circ, 0) = -\frac{\partial}{\partial q}(T - V)\Big|_{\substack{q=q^\circ \\ \dot q = 0}} = \frac{\partial V}{\partial q}(q^\circ) = 0$$

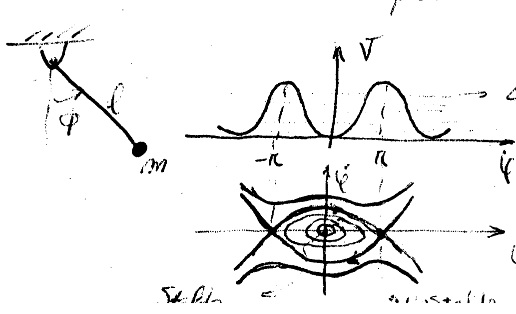
holds at equilibria for Conservative systems

Geometrically

Stability: q° is stable if for all small perturbations the resulting motion stays close to q°. $\forall \varepsilon > 0\ \exists \delta > 0$ Such that for all $|q(0) - q^\circ| \leq \delta$ we have $|q(t) - q^\circ| < \varepsilon$ for all $t \geq 0$

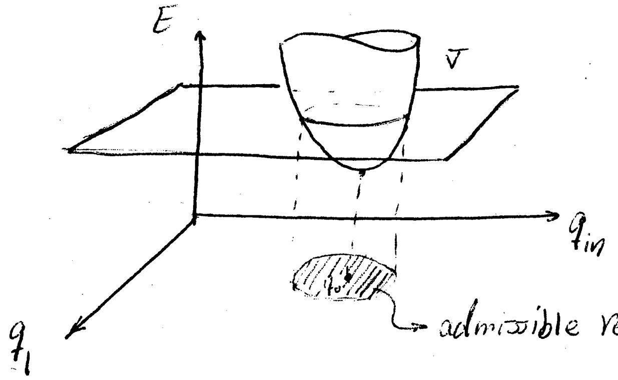
q° is unstable if not Stable, (i.e. there is at least one perturbation that grows)

Example



Stability Criterion (Dirichlet) in a Conservative System

an equilibrium q° is stable if and only if V has a ~~~~ (strict) local minimum.



$E = T + V = \text{Const}$

$T \geq 0$

E is slightly over $V(q^\circ)$

→ admissible region for trajectory $q(t)$ with q° in R

How do we find local minimum of V

- $\left.\frac{\partial V}{\partial q}\right|_{q=q^\circ} = 0$ (extremum point)
- $\left[\frac{\partial^2 V}{\partial q_i \, \partial q_j}\right]_{q=q^\circ}$ is positive definite

(Hessian matrix)

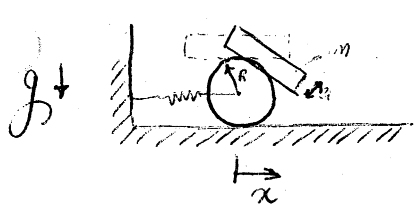
Sufficient and necessary Condition for the positive definiteness of a ~~System~~ Symmetric matrix:

1) all its eigenvalues are positive

2) $\begin{bmatrix} a_{11} & a_{12} & \\ a_{21} & a_{22} & \\ & & \end{bmatrix}$ $D_1 = \det[a_{11}]$ $D_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\det(D_i) > 0 \quad i = 1, \dots, n$

Example (written dynamics qual question 2004)

Rolling disk with tipping block, Constrained by spring



both objects roll without slip

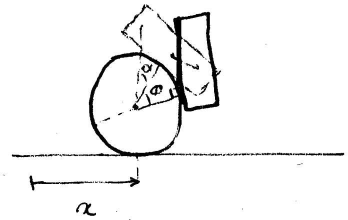
$\# DOF = 2 \times 3 - 2 \times 2 = 2$

- active forces are potential (gravity - spring)
- Constraint forces do not do work (rolling)

} → System is Conservative ⇓ Dirichlet theorem applies

Understand displacement at block by

- first rolling the disk to positive x with the block fixed to the disk
- then Superimpose the rolling of the disk to positive φ

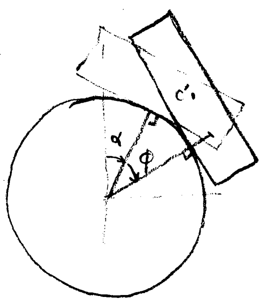


$\alpha = \frac{x}{R}$

$\sin(\alpha+\varphi)(R\varphi)$

$\cos(\alpha+\varphi)(R+\frac{a}{2})$

$y_c = R\varphi \sin(\frac{x}{R}+\varphi) + (R+\frac{a}{2})\cos(\frac{x}{R}+\varphi)$



$$V = V^{\text{spring}} + V^{\text{block}} + V^{\text{disk}}$$

$$= \frac{1}{2}kx^2 + mg\left[(R+\tfrac{a}{2})\cos(\tfrac{x}{R}+\varphi) + R\varphi\sin(\tfrac{x}{R}+\varphi)\right]$$

$$\begin{cases} \dfrac{\partial V}{\partial x} = kx + mg\left[\dfrac{2R+a}{2R}\sin(\tfrac{x}{R}+\varphi) + \varphi\cos(\tfrac{x}{R}+\varphi)\right] \\ \dfrac{\partial V}{\partial \varphi} = mg\left[-\dfrac{a}{2}\sin(\tfrac{x}{R}+\varphi) + R\varphi\cos(\tfrac{x}{R}+\varphi)\right] \end{cases}$$

Note: $\dfrac{\partial V}{\partial x}^{(0,0)} = 0$

$\dfrac{\partial V}{\partial \varphi}^{(0,0)} = 0$

$\Rightarrow (x,\varphi) = (0,0)$ is indeed equilibrium

Stability criterion (Dirichlet) in a Conservative System

For Stability: $\dfrac{\partial^2 V}{\partial x^2} = k + mg\left[\dfrac{2R+a}{2R^2}\cos(\tfrac{x}{R}+\varphi) - \dfrac{\varphi}{R}\sin(\tfrac{x}{R}+\varphi)\right]$

$$\frac{\partial^2 V}{\partial x\,\partial \varphi} = mg\left[-\frac{a}{2R}\cos(\tfrac{x}{R}+\varphi) - \varphi\sin(\tfrac{x}{R}+\varphi)\right]$$

$$\frac{\partial^2 V}{\partial \varphi^2} = mg\left[\frac{2R-a}{2}\cos(\tfrac{x}{R}+\varphi) - R\varphi\sin(\tfrac{x}{R}+\varphi)\right]$$

Hessian matrix of V at equil. $D^2V = \begin{bmatrix} K - mg\frac{2R+a}{2R^2} & -mg\frac{a}{2R} \\ \text{Sym.} & mg\frac{2R-a}{2} \end{bmatrix}$

D^2V is positive definition if and only if (1) $K > mg\dfrac{2R+a}{2R^2}$

(2) $\left[K - mg\dfrac{2R+a}{2R^2}\right] mg\dfrac{2R-a}{2} > \dfrac{m^2g^2a^2}{\ldots}$

if (1) holds, (2) can only hold if $R > a/2$, in which case (2) simplifies

$$(3) \quad K > mg\left[\frac{2R+a}{2R^2} + \frac{a^2}{2R^2(2R-a)}\right] = \frac{2mg}{2R-a}$$

thus (3) is stronger than (1), the final set of conditions for stability

$$R > \frac{a}{2}, \quad K > \frac{2mg}{2R-a}$$