2.016 Hydrodynamics

Prof. A.H. Techet

Introduction to basic principles of fluid mechanics

I. Flow Descriptions

1. Lagrangian (following the particle):

In rigid body mechanics the motion of a body is described in terms of the body's position in time. This body can be translating and possibly rotating, but not deforming. This description, following a particle in time, is a *Lagrangian* description, with velocity vector

$$\overline{V} = u\,\hat{i} + v\,\hat{j} + w\,\hat{z}\,. \tag{3.1}$$

Using the Lagrangian approach, we can describe a particle located at point $\overline{x_o} = (x_o, y_o, z_o)$ for some time $t = t_o$, such that the particle velocity is

$$\vec{V} = \frac{\partial\vec{x}}{\partial t}, \tag{3.2}$$

and particle acceleration is

$$\bar{a} = \frac{\partial\overline{V}}{\partial t}\,. \tag{3.3}$$

We can use Newton's Law of motion ($\vec{F} = m\vec{a}$) on the body to determine the acceleration and thus, the velocity and position. However, in fluid mechanics, it is difficult to track a single fluid particle. But in the lab we can observe many particles passing by one single location.

2. Eulerian (observing at one location):

In the lab, we can easily observe many particles passing a single location, and we can make measurements such as drag on a stationary model as fluid flows past. Thus it is useful to use the *Eulerian* description, or control volume approach, and describe the flow at every fixed point in space (x, y, z) as a function of time, t.

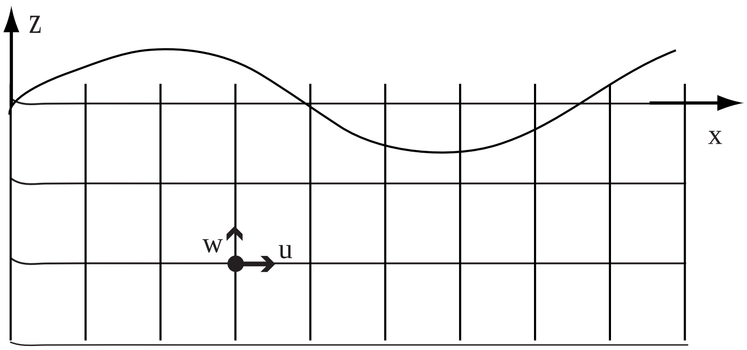


Figure 1: An *Eulerian* description gives a velocity vector at every point in x,y,z as a function of time.

In an Eulerian velocity field, velocity is a function of the position vector and time, $\vec{V}(\vec{x},t)$. For example:

$$\vec{V}(\vec{x},t) = 6tx^2\hat{i} + 3zy\hat{j} + 10xyt^2\hat{z}$$

3. Reynolds Transport Theorem (the link between the two views):

In order to apply Newton's Laws of motion to a control volume, we need to be able to link the control volume view to the motion of fluid particles. To do this, we use the Reynolds Transport Theorem, which you'll derive in graduate fluids classes, like 2.25. Suffice it to say that the theorem exists. For this class, we'll use control volumes to describe fluid motion.

4. Description of Motion:

Streamlines: Line everywhere tangent to velocity (Eulerian) (No velocity exists perpendicular to the streamline!)

Streaklines: instantaneous loci of all fluid particles that pass through a given point x_o.

Particle Pathlines: Trajectory of fluid particles ("more" lagrangian)

In ***steady flow*** stream, streak, and pathlines are identical!! (Steady flow has no time dependence.)

II. Governing Laws

The governing laws of fluid motion can be derived using a control volume approach. This is equivalent to a "fluidic black box" where all we know about the flow is what is going in and what is coming out of the control volume: mass, momentum, and energy. The control volume (CV) can be fixed or move with the fluid. For simplicity it is often ideal to fix the CV, but this does not always provide the easiest solution in all cases. For most of this class the CV will be fixed.

When analyzing a control volume problem there are three laws that are always true:

1. Conservation of Mass
2. Conservation of Momentum
3. Conservation of Energy

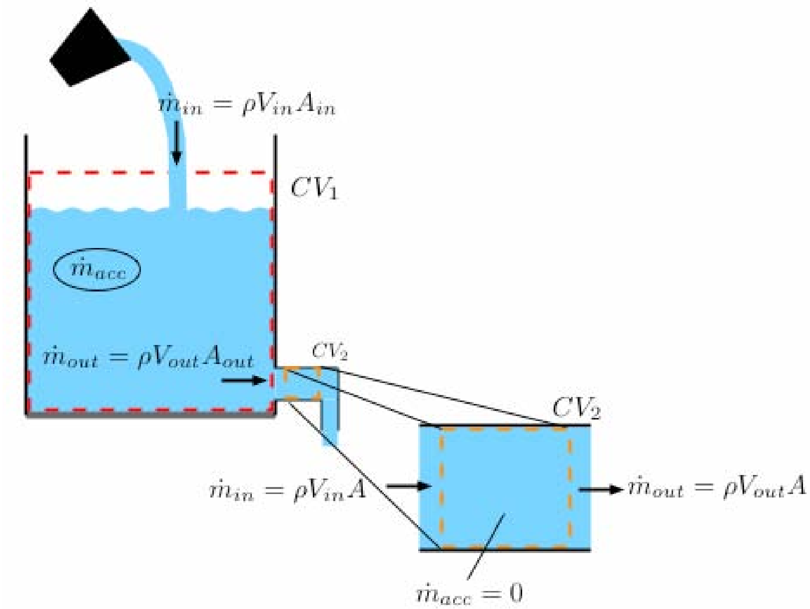
1. Conservation of Mass:

Basic fluid mechanics laws dictate that mass is conserved within a *control volume* for constant density fluids. *Thus the total mass entering the control volume must equal the total mass exiting the control volume plus the mass accumulating within the control volume.*

mass in – mass out = mass accumulating

$$\dot{m}_{in} - \dot{m}_{out} = \dot{m}_{acc} \tag{3.4}$$

Let us consider three cases:



Case I:

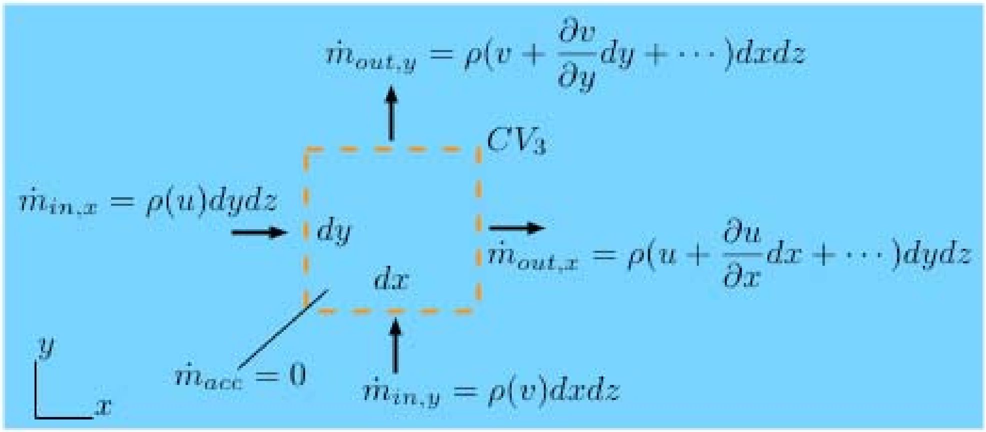
In control volume 1, water is poured into a tub with a drain. The mass flow rate into the tub is $\dot{m}_{in} = \rho V_{in} A_{in}$. Similarly, the mass flow rate out of the tub is $\dot{m}_{out} = \rho V_{out} A_{out}$. If the mass flow rate in is greater than the mass flow rate out, water will accumulate in the tub. If the mass flow rate out is greater, then the tub will drain. If the two are equal, then no water will accumulate in the tub. Think about it next time you're in the shower!

Case II:

Control Volume 2 is a section of a pipe full of water. Since the CV is full, the only way more mass can accumulate is if it becomes denser. Remember, in this class, we treat water as incompressible, so the density cannot change and we never have mass accumulating in a full control volume. Thus, the mass flow rate in equals the mass flow rate out. Furthermore, since CV_2 is drawn in a pipe of constant area, then the velocity in must equal the velocity out. For an incompressible fluid, there is no change in velocity through a pipe of constant area!

Case III:

Let's now consider a general control volume immersed in a fluid:



We can write a 2D mass balance equation for the fluid entering and exiting the control volume.

$$\dot{m}_{in} - \dot{m}_{out} = \dot{m}_{acc} \tag{3.5}$$

$$\rho u dydz + \rho v dxdz - \rho(u + \frac{\partial u}{\partial x}dx)dydz - \rho(v + \frac{\partial v}{\partial y}dy)dxdz = 0 \tag{3.6}$$

$$-\rho\frac{\partial u}{\partial x}dxdydz - \rho\frac{\partial v}{\partial y}dydxdz = 0 \tag{3.7}$$

Which simplifies to,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3.8}$$

In three dimensions, the derivation is the same, and we have:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{3.9}$$

or in vector notation, recalling the gradient operator: $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$, we have

$$\nabla \cdot \vec{V} = 0 \tag{3.10}$$

2. Conservation of Momentum:

Newton's second law is simply the *law of conservation of momentum*.

It states that *the time rate of change of momentum of a system of particles is equal to the sum of external forces acting on that body*.

$$\Sigma \mathbf{F}_i = \frac{d}{dt}\{M\mathbf{V}\} \tag{3.11}$$

where $M = \rho\delta x\delta z$ is the mass of the fluid parcel (in two dimensions, ie mass per unit length) and $M\mathbf{V}$ is the linear momentum of the system (**V** is the velocity vector). Since the fluid density is constant, the time-rate of change of linear momentum can be written as

$$\frac{d}{dt}\{M\mathbf{V}\} = \rho\delta x\delta z\frac{d\mathbf{V}}{dt}. \tag{3.12}$$

The rate of change of velocity of the fluid parcel can be found, for small δt, as

$$\frac{d\mathbf{V}}{dt} = \lim_{\delta t\to 0}\frac{1}{\delta t}\left\{\mathbf{V}(x+\delta x_p, z+\delta z_p, t+\delta t_p) - \mathbf{V}(x,z,t)\right\} \tag{3.13}$$

We can substitute, $\delta x_p = u\delta t$, and, $\delta z_p = w\delta t$, into equation (3.13) and cancel terms to arrive at a more familiar form of the momentum equation.

The total derivative of the velocity is written as:

$$\frac{D\mathbf{V}}{Dt} = \frac{\partial\mathbf{V}}{\partial t} + \frac{\partial\mathbf{V}}{\partial x}u + \frac{\partial\mathbf{V}}{\partial z}w \tag{3.14}$$

which can be simplified using the vector identity,

$$\mathbf{V}\cdot\nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \tag{3.15}$$

The total (material) derivative of the velocity is the sum of the conventional acceleration, $\frac{\partial V}{\partial t}$, and the advection term, $(\mathbf{V}\cdot\nabla)\mathbf{V}$:

$$\frac{D\mathbf{V}}{Dt}=\frac{\partial\mathbf{V}}{\partial t}+(\mathbf{V}\cdot\nabla)\mathbf{V}. \tag{3.16}$$

Finally, the momentum equation, from equation (3.11), can be rewritten in two dimensions as

$$\Sigma\mathbf{F_i}=\rho\frac{D\mathbf{V}}{Dt}\delta x\delta z\,. \tag{3.17}$$

3. Forces

The LHS of equation (3.17) is the *sum of the forces acting on the control volume*. Contributions from gravity and pressure both play a role in this term as well as any applied external forces. These forces are found as follows:

1. Force on a fluid volume due to gravity:

$$\mathbf{F_g}=-(\rho g\delta x\delta z)\hat{k} \tag{3.18}$$

2. Pressure Forces due: Force due to pressure is simply pressure times the surface area it acts on

$$F_P=P\cdot A\,. \tag{3.19}$$

The pressure force in x-direction is

$$\mathbf{F_{Px}}=\left(p+\frac{1}{2}\frac{\partial p}{\partial z}\delta z\right)\delta z-\left(p+\frac{1}{2}\frac{\partial p}{\partial z}\delta z+\frac{\partial p}{\partial x}\delta x\right)\delta z=-\frac{\partial p}{\partial x}\delta x\delta z \tag{3.20}$$

and the pressure force in z-direction is

$$\mathbf{F_{Pz}}=-\frac{\partial p}{\partial z}\delta x\delta z\,. \tag{3.21}$$

Thus the total pressure force in two dimensions is

$$F_P = -(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial z})\delta x \delta z = -\nabla p \delta x \delta z. \tag{3.22}$$

4. Euler Equation

Substituting relations (3.18) and (3.22) for the gravity and pressure forces acting on the body, into the momentum equation (3.17) we arrive at

$$\rho \left\{ \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} \right\} \delta x \delta z = (-\rho g \delta x \delta z)\hat{k} - \nabla p \delta x \delta z \tag{3.23}$$

for any $\delta x, \delta z$. The final result is the Euler equation in vector form:

$$\rho \left\{ \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} \right\} = -\rho g \hat{k} - \nabla p. \tag{3.24}$$

We can further manipulate this equation with the vector identity

$$(\mathbf{V} \cdot \nabla)\mathbf{V} = \tfrac{1}{2}\nabla(\mathbf{V} \cdot \mathbf{V}), \tag{3.25}$$

such that the Euler equation becomes

$$\rho \left\{ \frac{\partial \mathbf{V}}{\partial t} + \frac{1}{2}\nabla(\mathbf{V} \cdot \mathbf{V}) \right\} = -\rho g \hat{k} - \nabla p. \tag{3.26}$$

5. Bernoulli's Equation:

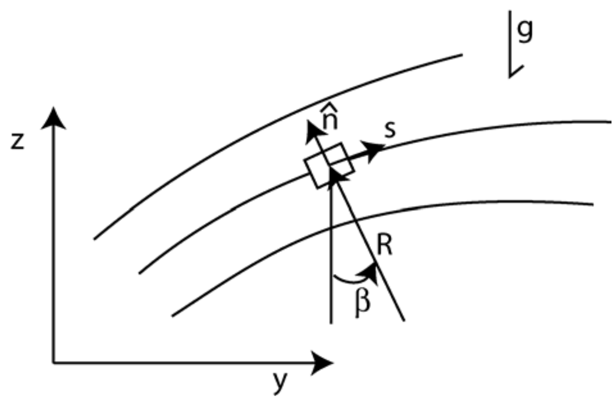
Application of *Newton's Second Law along a streamline*:

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \rho g z_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \rho g z_2 = C \tag{3.27}$$

Assuming the following conditions:

1) Points 1 and 2 are on the same streamline!
2) Fluid density is constant
3) Flow is steady: $\frac{dV}{dt} = 0$ (no time dependence or turbulence)
4) Fluid is "inviscid" or can be approximated as inviscid. No frictional effects
5) No Work Added!

We can derive this through a Lagrangian derivation:



Looking at a small elemental volume along a streamline $d\forall = dn\, ds\, dx$ (dx is the depth into the paper).

Fluid weight in the (-z) direction.:

$$\rho g\, dn\, ds\, dx \tag{3.28}$$

Component of weight acting in the s-direction:

$$-\rho g \sin\beta\, dn\, ds\, dx \tag{3.29}$$

Where $\sin\beta = \frac{dz}{ds}$ so that the weight in the s-direction is:

$$-\rho g\,\frac{dz}{ds}\,dn\;ds\;dx\,. \qquad (3.30)$$

The force due to pressure in the s-direction is found similarly:

$$F_s = \left(p - \frac{\partial p}{\partial s}\right) dn\;dx - \left(p + \frac{\partial p}{\partial s}\,\frac{ds}{2}\right) dn\;dx = \left(-\frac{\partial p}{\partial s} - \rho g\,\frac{\partial z}{\partial s}\right) dn\;ds\;dx \qquad (3.31)$$

The force accelerates the fluid along the streamline such that the rate of change in momentum, per unit volume, is

$$\rho\left(\frac{V + \frac{\partial V}{\partial s}ds - V}{dt}\right) = \rho V\,\frac{\partial V}{\partial s} \qquad (3.32)$$

where $V = \dfrac{\partial s}{\partial t}$.

So Euler's equation in one dimension along a streamline becomes:

$$\boxed{\rho V\,\frac{\partial V}{\partial s} + \frac{\partial p}{\partial s} + \rho g\,\frac{\partial z}{\partial s} = 0}. \qquad (3.33)$$

Change in Pressure along a streamline: $dp = \dfrac{\partial p}{\partial s}ds$

Change in Velocity along a streamline: $dV = \dfrac{\partial V}{\partial s}ds$

Change in height along a streamline: $dz = \dfrac{\partial z}{\partial s}ds$

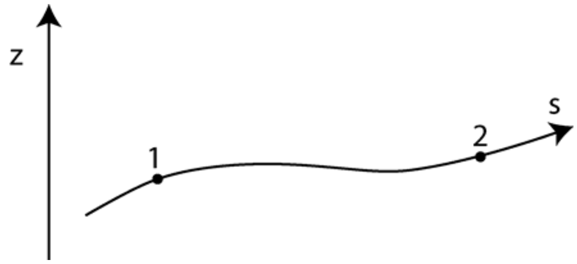
Multiplying equation 23 through by *ds* gives us

$$\rho V\;dV + dp + \rho g\;dz = 0 \qquad (3.34)$$

$$\frac{dp}{\rho} + V\;dV + g\;dz = 0 \qquad (3.35)$$

If density is constant along the streamline then we can integrate along the streamline to get:

$$\frac{p}{\rho} + \frac{1}{2}V^2 + g\ z = C$$



Along a streamline Bernoulli's equation relates pressure, height and velocity at two points:

$$\frac{p_1}{\rho} + \frac{1}{2}V_1^2 + g\ z_1 = \frac{p_2}{\rho} + \frac{1}{2}V_2^2 + g\ z_2 = C \tag{3.36}$$

This equation also assumes that NO additional heat or work is added to the system along the streamline.

6. Irrotational flow

An *irrotational flow* is defined as a flow for which each and every fluid particle is not rotating. Mathematically speaking, the curl of the velocity is identically zero.

$$\omega = \nabla \times \mathbf{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = 0 \tag{3.37}$$

$$\omega = \hat{i}(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}) + \hat{j}(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}) + \hat{k}(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) = 0. \tag{3.38}$$

For 2D flow this reduces to $\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}$.

