2.004 Dynamics and Control II
Spring 2008

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Lecture 25[1]

Reading:

- Nise: Chap. 6

1 System Stability

gravitational force acts to
restore a displaced ball to
its equilibrium position

gravitational force acts to
move a displaced ball away
from its equilibrium position

no forces act to move a
displaced ball

(a) a stable system   (b) an unstable system   (c) a neutrally stable system
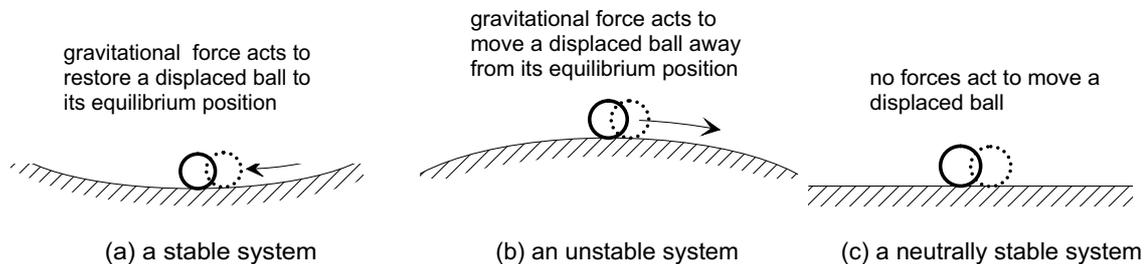
The figure above illustrates three stability conditions using a rolling ball on an undulating surface as a graphic example. Assume that the horizontal position and velocity of the ball are a pair of state variables describing this system. In a concave region of the surface, as shown in in (a) the base of the hollow is an equilibrium point. If the ball is displaced a small distance from this position and released it oscillates but ultimately returns to its rest position at the base as it loses energy due to frictional losses; this is therefore a stable equilibrium point. (Without energy dissipation the ball would roll back and forth forever and exhibit neutral, or marginal, stability.) On a convex portion of the surface, as shown in (b), the ball is in equilibrium if placed *exactly* at the top of the surface, but if it is displaced an infinitesimal distance to either side the net gravitational force acting on the ball causes it to roll down the surface and never to return to the equilibrium point. This equilibrium point is therefore unstable. If the ball is displaced along the flat portion of the surface, as shown in (c) it neither moves away nor returns; the flat portion represents a neutrally stable equilibrium region.

There are many formal definitions of stability of a system.

- From the solution of the linear differential equation

$$y(t) = y_h(t) + y_p(t)$$

   a system is defined to be *asymptotically stable*, if for all initial conditions, the steady-state homogeneous response returns to zero. If the initial condition response increases

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without bound the system is said to be *unstable*, and if the steady-state response is constant or purely oscillatory the system is defined to be *marginally stable*.

- From the forced response, a system is defined to be *bounded-input bounded-output* (BIBO) stable only if every bounded input produces a bounded output.

These two definitions are not the same. A system

$$G(s) = \frac{1}{s^2 + \omega_n^s}$$

has a pair of imaginary poles at $s = \pm j\omega_n$ and will exhibit a purely oscillatory response from any finite initial conditions. On this basis the system is therefore marginally stable. On the other hand, if excited by an input $\sin(\omega_n t)$ the response amplitude will tend to infinity, therefore the system is BIBO unstable.

From our previous work on linear system response, we have seen that if any system pole lies in the r.h. *s*-plane the system response will grow without bound as a result of any finite initial condition on the output of its derivatives. A fundamental definition of system stability is therefore

A linear system is defined to be stable only if all of its poles have negative real parts. A system with one or more poles that are purely imaginary are defined to be marginally stable.

The following figure shows a system with r.h. plane poles and the response components that result.
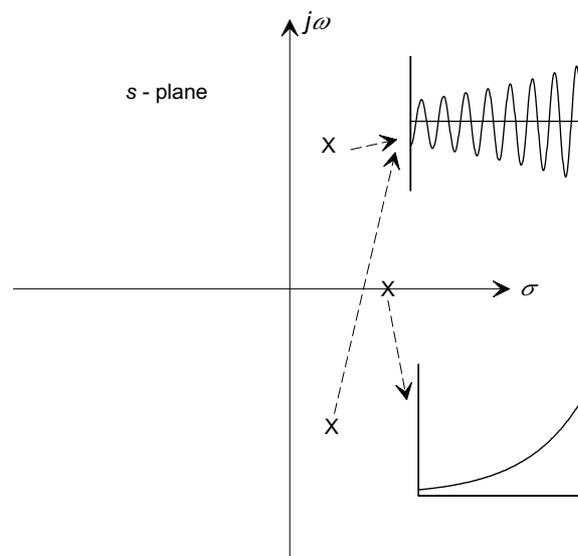
■ Example 1

Determine the stability of the two systems

$$G_1(s) = \frac{15}{s^3 + 2s^2 + 5s + 15} \quad \text{and} \quad G_2(s) = \frac{5}{s^3 + 2s^2 + 5s + 5}.$$

MATLAB's `roots()` function was used to find the roots of the characteristic equation for each system, to determine the system poles:

1. For $G_1(s)$: `roots([1 2 5 15])` gave the system poles as $s = -2.4357, s = 0.2268 \pm j2.462$.

2. For $G_2(s)$: `roots([1 2 5 5])` gave the system poles as $s = -1.233, s = -0.384 \pm j1.977$.

System $G_1(s)$ has a pair of complex conjugate poles in the r.h. plane and is therefore *unstable*. System $G_2(s)$ has all of its poles in the l.h. plane and is therefore *stable*.

The task of determining system stability is therefore that of determining the roots of the characteristic equation $D(s) = 0$, or determining whether any root has a positive real part. Before the advent of numerical software packages, such as MATLAB, this was a very difficult problem.

2   The Routh-Hurwitz Criterion

The Routh-Hurwitz criterion is a method for determining the presence the number of roots of a polynomial with positive real parts.

- It is based on the coefficients of the polynomial.

- It involves placing the coefficients into a 2-D *Routh array*, and then systematically extending the array by creating new rows. When the array is complete, the number of r.h. plane roots can be found by inspection.

The method has two basic steps:

Step 1: A preliminary step is the following

> **Given a polynomial, a sufficient condition for the existence of at least one r.h. plane root is that there is a sign change in the coefficients.**

For example the system

$$G(s) = \frac{27}{s^5 + 47s^4 - 23s^3 + 19s^2 + 6s + 9}$$

is unstable because there is a sign change in the coefficients of D(s).

Put another way, for a system to be stable there must be no sign changes in the coefficients of the denominator polynomial $D(s)$.

Step 2: The Routh array is created and examined, as described below.

2.1 The Routh Array

Let the denominator polynomial be

$$D(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0$$

the initial step is to place the coefficients in the first two rows of the array:

s^n	a_n	a_{n-2}	a_{n-4}	$\ldots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\ldots$

Additional rows are then created, based on the two rows above. For example, let the next row be labelled s^{n-2} and s^{n-3} with entries b_i and c_i:

s^n	a_n	a_{n-2}	a_{n-4}	$\ldots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\ldots$
s^{n-2}	b_1	b_2	b_3	$\ldots$
s^{n-3}	c_1	c_2	c_3	$\ldots$

- Each row is constructed from left to right, until all further entries are zero.

- Each entry is based on entries in the two rows above, using values from (i) the first (leftmost) column, and the column to its right.

- The entries us a 2×2 determinant.

In the above example the b_i are computed from the rows labelled s_n and s_{n-1}:

$$b_1 = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}, \quad b_2 = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}, \quad b_3 = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-6} \\ a_{n-1} & a_{n-7} \end{vmatrix}, \quad \ldots$$

and the c_i are computed from the rows labelled s^{n-1} and s^{n-2}:

$$c_1 = \frac{1}{b_1} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix}, \quad c_2 = -\frac{1}{b_1} \begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix}, \quad c_3 = -\frac{1}{b_1} \begin{vmatrix} a_{n-1} & a_{n-7} \\ b1 & b_4 \end{vmatrix}, \quad \ldots$$

The process is repeated until the row that will be labelled s^0. Note that in each determinant the left column is taken from the *left* column of the Routh array, the second column is taken from the *column to the right* in the Routh array.

When the table is complete, **the number of r.h. plane roots is the number of sign changes in the first column**.

■ Example 2

Find the number of r.h. plane poles in the system

$$G(s) = \frac{37}{s^3 + s^2 + 2s + 24}$$

The first two rows of the Routh array are taken directly from $D(s)$:

| s^3 | 1 | 2 | 0 |
| s^2 | 1 | 24 | 0 |

where an additional zero has been added at the end of each row. Then construct the s^1 row:

$$b_1 = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} = -\frac{1}{1} \begin{vmatrix} 1 & 2 \\ 1 & 24 \end{vmatrix} = -22$$

$$b_2 = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix} = -\frac{1}{1} \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

and the Routh array, with the new row, becomes

s^3	1	2	0
s^2	1	24	0
s^1	-22	0	0

The s^0 row is now computed:

$$c_1 = -\frac{1}{b_1} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix} = -\frac{1}{22} \begin{vmatrix} 1 & 24 \\ -22 & 0 \end{vmatrix} = 24$$

$$c_2 = -\frac{1}{b_1} \begin{vmatrix} an-1 & a_{n-3} \\ b_1 & b_3 \end{vmatrix} = -\frac{1}{22} \begin{vmatrix} 1 & 0 \\ -22 & 0 \end{vmatrix} = 0$$

and the completed Routh array is

s^3	1	2	0
s^2	1	24	0
s^1	-22	0	
s^0	24	0	

The next step is to examine the first column; in this case we note there are *two* sign changes $1 \to -22 \to 24$, and we conclude that there are therefore *two* unstable r.h. plane poles in this system.

In fact, for this system $D(s) = (s + 3)(s - 1 + j\sqrt{7})(s - 1 - j\sqrt{7})$.

■ Example 3

Use the Routh-Hurwitz method to develop a simple test for the stability of a third-order system.

Let the denominator polynomial be

$$D(s) = a^3 s^3 + a_2 s^2 + a_1 s + a_0$$

and assume that all coefficients are positive. The complete Routh array is

s^3	a_3	a_1	0
s^2	a_2	a_0	0
s^1	$-(a_3 a_0 - a_2 a_1)/a_1$	0	
s^0	a_0	0	

and to ensure stability we require all terms in the first column to be positive, that is

$$a_1 a_2 > a_0 a_3$$

In other words a third-order system will be stable if the product of the inner coefficients is greater than the product of the outer coefficients.

Note: The Routh-Hurwitz method has some special cases to deal with situations when zeros appear in the first column. See Nise, Chapter 6, for details.