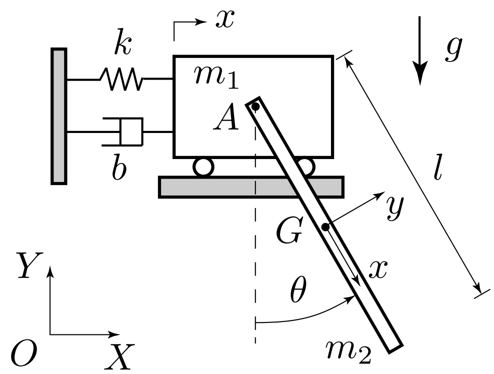
2.003SC
Recitation 7 Notes: Equations of Motion for Cart & Pendulum (Lagrange)

Cart and Pendulum - Problem Statement

A cart and pendulum, shown below, consists of a cart of mass, m_1 , moving on a horizontal surface, acted upon by a spring and damper with constants k and b , respectively. From the cart is suspended a pendulum consisting of a uniform rod of length, l , and mass, m_2 , pivoting about point A .



KINEMATICS: Write

- an expression for the linear velocity of point G, i.e. $\underline{v}_G$.
- an expression for the linear acceleration of point G, i.e. $\underline{a}_G$.

Cart and Pendulum - Kinematics

$\underline{v}_G = \dot{x}\hat{I} + \frac{l}{2}\dot{\theta}\hat{j}$

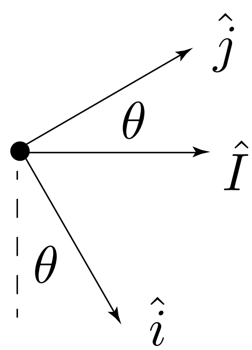
$\underline{a}_G = \ddot{x}\hat{I} + \frac{l}{2}\ddot{\theta}\hat{j} + \frac{l}{2}\dot{\theta}\frac{d}{dt}\hat{j}$ Recall that $\frac{d}{dt}\hat{j} = -\dot{\theta}\hat{i}$

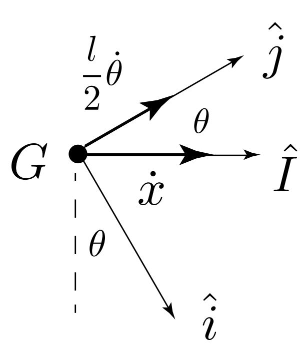
$\underline{a}_G = \ddot{x}\hat{I} - \frac{l}{2}\dot{\theta}^2\hat{i} + \frac{l}{2}\ddot{\theta}\hat{j}$

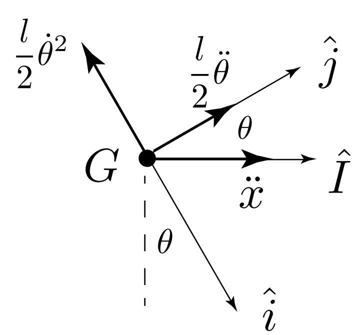
In terms of the $\hat{i}$ and $\hat{j}$ unit vectors,

$$\underline{a}_G = (\ddot{x}sin\theta - \frac{l}{2}\dot{\theta}^2)\hat{i} + (\ddot{x}cos\theta + \frac{l}{2}\ddot{\theta})\hat{j} \quad (1)$$

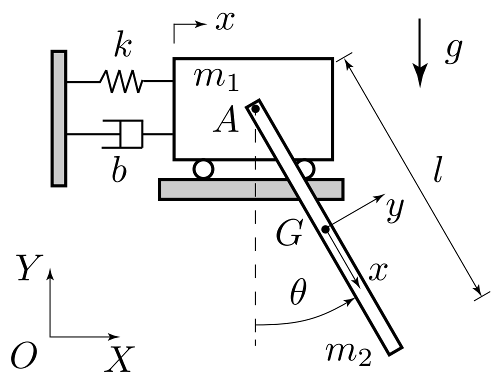
The figures below show the unit vectors, as well as point G's velocity and acceleration components.





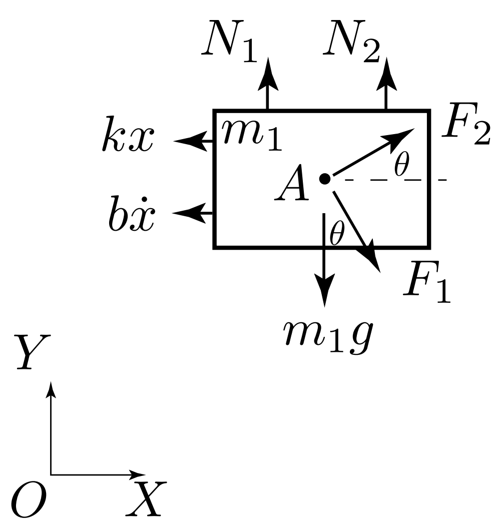


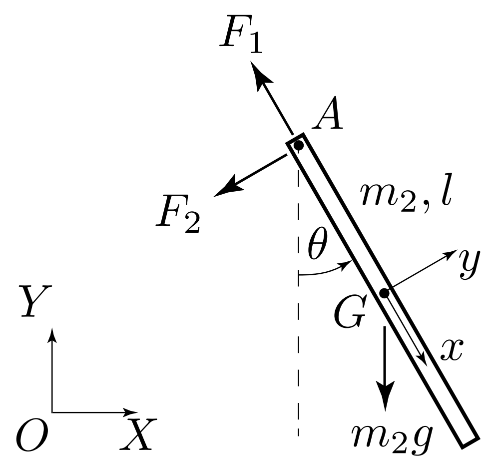
Cart and Pendulum - Free Body Diagrams



Draw the system's free body diagrams.

Cart and Pendulum - Free Body Diagrams





Cart and Pendulum - Newton's Laws

Summing forces on mass m_2 in the $\hat{i}$ and $\hat{j}$ -directions,

$$-F_1 + m_2 g cos\theta = m_2 a_x^G \tag{2}$$

$$-F_2 - m_2 g sin\theta = m_2 a_y^G \tag{3}$$

Equations (2) and (3) can be rewritten to isolate F_1 and F_2 , and the acceleration components from equation (1) substituted.

$$F_1 = m_2 g cos\theta - m_2 a_x^G = m_2 g cos\theta - m_2(\ddot{x} sin\theta - \frac{l}{2}\dot{\theta}^2)$$

$$F_2 = -m_2 g sin\theta - m_2 a_y^G = -m_2 g sin\theta - m_2(\ddot{x} cos\theta + \frac{l}{2}\ddot{\theta})$$

Summing forces on mass m_1 in the $\hat{I}$ -direction,

$$-kx - b\dot{x} + F_1 sin\theta + F_2 cos\theta = m_1 \ddot{x} \tag{4}$$

Summing torques **ABOUT POINT G** on mass m_2 in the $+\theta$ -direction,

$$F_2 \frac{l}{2} = \frac{m_2 l^2}{12}\ddot{\theta} \tag{5}$$

Substituting (2) and (3) into equations (4) and (5), we obtain,

$$-kx - b\dot{x} + [m_2 g cos\theta - m_1(\ddot{x} sin\theta - \frac{l}{2}\dot{\theta}^2)]sin\theta + [-m_2 g sin\theta - m_1(\ddot{x} cos\theta + \frac{l}{2}\ddot{\theta})]cos\theta = m_1\ddot{x}$$

$$[-m_2 g sin\theta - m_1(\ddot{x} cos\theta + \frac{l}{2}\ddot{\theta})]\frac{l}{2} = \frac{m_2 l^2}{12}\ddot{\theta}$$

which (eventually) reduce to

$$(m_1 + m_2)\ddot{x} + b\dot{x} + kx + \frac{m_2 l}{2}\ddot{\theta} cos\theta - \frac{m_2 l}{2}\dot{\theta}^2 sin\theta = 0 \tag{6}$$

and

$$(\frac{m_2 l^2}{12} + \frac{m_2 l^2}{4})\ddot{\theta} + \frac{m_2 l}{2}\ddot{x} cos\theta + m_2 g \frac{l}{2} sin\theta = 0 \tag{7}$$

Summing torques about point A...

$$\Sigma^A \underline{\tau} = \frac{d^A \underline{H}}{dt} + \underline{v}_A \times \underline{P}_G$$

$$\underline{P}_G = m\underline{v}_G = m(\dot{x}\hat{I} + \frac{l}{2}\dot{\theta}\hat{j})$$

$$\Sigma^A \underline{\tau} = \frac{d}{dt}[^G\underline{H} + \underline{r} \times \underline{P}_G] + \underline{v}_A \times \underline{P}_G$$

$$\Sigma^A \underline{\tau} = \frac{d}{dt}[\frac{m_2 l^2}{12}\dot{\theta} + \frac{l}{2}\hat{i} \times m_2(\dot{x}\hat{I} + \frac{l}{2}\dot{\theta}\hat{j})] + \dot{x}\hat{I} \times m_2(\dot{x}\hat{I} + \frac{l}{2}\dot{\theta}\hat{j})$$

$$-m_2 g\frac{l}{2}sin\theta = \frac{d}{dt}[\frac{m_2 l^2}{12}\dot{\theta} + \frac{m_2 l}{2}\dot{x}cos\theta + \frac{m_2 l^2}{4}\dot{\theta}] + \frac{m_2 l}{2}\dot{x}\dot{\theta}sin\theta$$

which (eventually) reduces to equation (7) above,

$$(\frac{m_2 l^2}{12} + \frac{m_2 l^2}{4})\ddot{\theta} + \frac{m_2 l}{2}\ddot{x}cos\theta + m_2 g\frac{l}{2}sin\theta = 0$$

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