2.003J/1.053J Dynamics and Control I, Spring 2007
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Lecture 13

Lagrangian Dynamics: Generalized Coordinates and Forces

Lecture Outline

Solve one problem by Newtonian and Lagrangian Methods.
"Lagrangian approach is simple but devoid of insight."
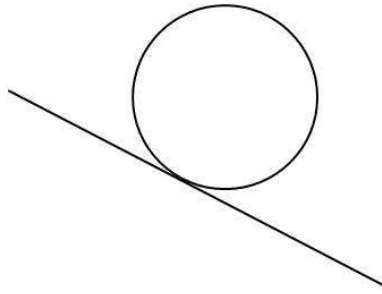Both methods can be used to derive equations of motion.

Figure 1: Wheel on an incline. Figure by MIT OCW.

1. Solve a well-known problem by Newton's method: Wheel down incline
2. Critique Solution
3. Present Lagrange Equations
4. Solve well-known problem by Lagrange's Method
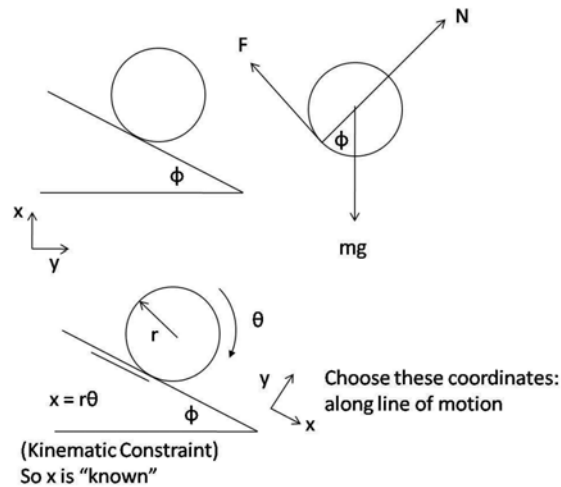
Example: Wheel Rolling Down Incline

Figure 2: Free body and kinematic diagrams of wheel rolling down incline. The wheel is subject to a normal force, N, a frictional force, F, and a gravitational force, mg. Figure by MIT OCW.

What is the acceleration?

F = friction
1 degree of freedom, θ.

Newton's Method

3 unknowns: $\underline{N}$, $\underline{F}$, and θ
Equations

$$\sum \underline{F}_x = m\underline{a}_x \qquad \sum \underline{F}_y = m\underline{a}_y$$
$$\sum \underline{\tau} = I\underline{\alpha}$$

x-direction:

$$-F + mg\sin\phi = mr\ddot{\theta}$$

y-direction:

$$N - mg\cos\phi = 0$$

Torques about center of wheel:

$$Fr = \frac{1}{2}mr^2\ddot{\theta}$$

Eliminate F:

$$mgr\sin\phi = \frac{3}{2}mr^2\ddot{\theta}$$

$$\boxed{\ddot{\theta} = \frac{2}{3}\frac{g}{r}\sin\phi}$$

Critique of Newton's Method

Critique:
   Did the reaction forces do any work? Did the reaction force move?
   N did not move. F applied but point did not move at point of contact.
   No work done by those forces.

1. Had to explicitly deal with non-contributing forces. Newton's Laws are derived in cartesian coordinates, but problem's answer was not in cartesian.
2. Newton's laws are written in Cartesian coordinates, but we really care about configurational parameters.
3. Had to write several kinematic constraints to reduce unknowns x and y.

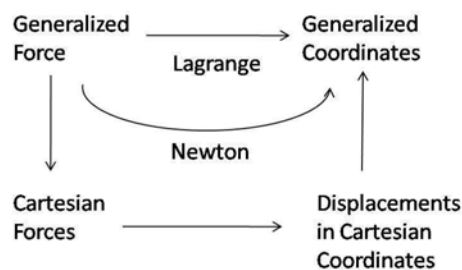## Schematic of Newton versus Lagrangian Approach

Figure 3: Diagram depicting the differences between Newtonian and Lagrangian approach. Figure by MIT OCW.
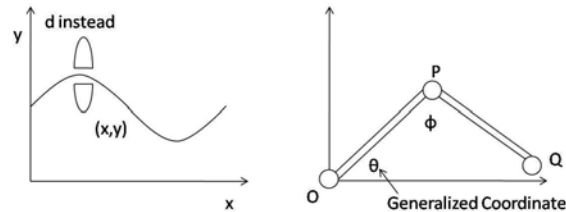
Generalized Coordinates

Figure 4: *Left.* Generalized coordinate for bead on wire. For a bead moving along a wire, one generalized coordinate, the distance along the wire, can be used to describe the position instead of the two Cartesian coordinates x and y. *Right.* Generalized coordinate for a two beam system. In the system of connected bars OP and PQ driven by a motor at Q, there is only one degree of freedom. Thus, an angular coordinate such as theta, angle of OP with the x-axis, or phi, angle between OP and PQ, completely describes the coordinates of the rigid body. If one was not using generalized coordinates, one would need x,y-coordinates at points O, P, and Q. Figure by MIT OCW.

Only 1 degree of freedom in both cases

Newton: x, y for each of points O, P, and Q. ϕ would also work. θ is a generalized coordinate; it naturally, completely, and independently describe coordinates of the rigid body.

The generalized coordinates of a system (of particles or rigid body or rigid bodies) is the natural, minimal, complete set of parameters by which you can completely specify the configuration of that system.

Lagrange Method

We have already seen a generalized force. Where? Torque.
Newton:

$$F = ma$$

coordinates = # degrees of freedom

Lagrange:

$$Q_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}$$

Q_i: generalized force
L: Lagrangian
q_i: generalized coordinates

$$T + V = \text{ Constant}$$

$$L = T - V$$

1 equation per coordinate
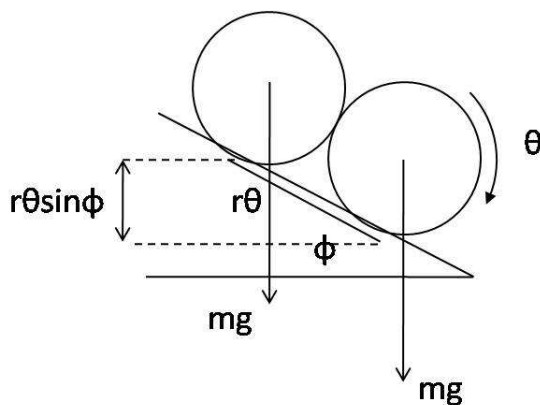No reaction forces appear

Application of Lagrange Method

Figure 5: Wheel rolls down incline. The wheel rotates by an angle, θ and traverses a distance of $r\theta$. Figure by MIT OCW.

Generalized Coordinate: $\theta = q_1$
Generalized Force: $Q_1 = 0$

$$T = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 = \frac{1}{2}\left(\frac{1}{2}mr^2\right)\dot{q}_1^2 + \frac{1}{2}mr^2\dot{q}_1^2$$

$$V = -mgr\theta\sin\phi = -mgrq_1\sin\phi$$

Gravity accounted for by V
Ignore non-contributing forces
   1. Internal rigid body forces
   2. Reaction forces across slipping surfaces
   3. Rolling forces

$$L = T - V = \frac{3}{4}mr^2\dot{q}_1^2 + mgrq_1 \sin\phi$$

$$\frac{\partial L}{\partial \dot{q}_1} = \frac{3}{2}mr^2\dot{q}_1$$

$$\frac{\partial L}{\partial q_1} = mgr\sin\phi$$

$$\ddot{q}_1 = \ddot{\theta} = \frac{2}{3}g\sin\phi$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) = \frac{3}{2}mr^2\ddot{q}_1$$

Non-Contributing Force Comments

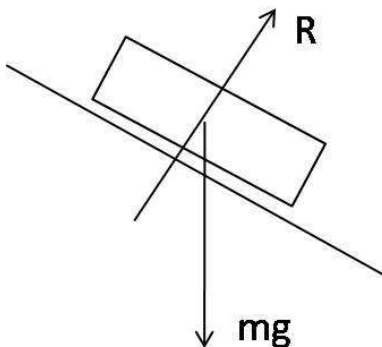Lagrange leads to math for 2 and 3 degrees of freedom problems.

Figure 6: Block sliding along incline. Two forces are present, a reaction force, R, and a gravitational force, mg. Figure by MIT OCW.
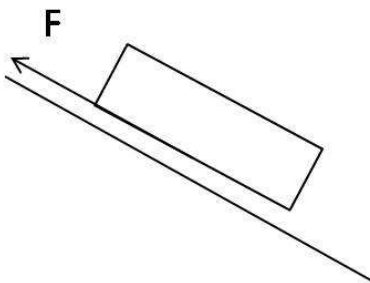
R can be ignored if frictionless

Figure 7: Free body diagram of block sliding along incline. F cannot be ignored if friction did work. Figure by MIT OCW.

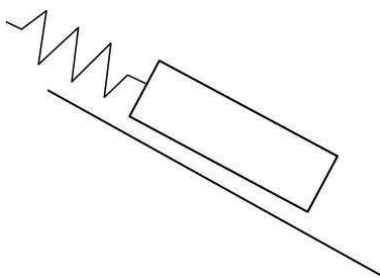F cannot be ingored (friction did work)

Figure 8: Free body diagram of block sliding along incline with attached spring. Spring contributes because spring lengthens. Figure by MIT OCW.

Spring contributes because spring lengthens.

Generalized Force Definition

Generalized Force:

$$Q_i = \sum_{j=1}^{m_{FORCES}} \underline{F}_j \frac{\partial \underline{r}_j}{\partial q_i}$$

Take every force and dot product with tangent direction.
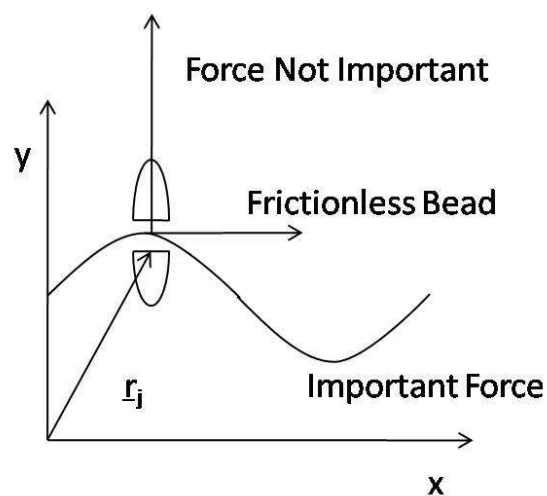Only consider forces in the admissible diections of motion.

Figure 9: Frictionless bead on a wire. Figure by MIT OCW.

Closing Comments

Lagrange is used when forces are all conservative. We can also use Lagrange when there is friction, for example, but then the equations involve more math. Lagrange works best for systems where all contributing forces are conservative.

Notice that Lagrange ignores noncontributing forces; the coordinates are natural instead of Cartesian, and it was not necessary to supply additional equations as kinematic constraints, so all of the critiques of Newton were answered by Lagrange.