3|31. Cameron's Talk.

<u>Defn</u> Given Γ ternary relation (write $a \underset{C}{\downarrow}^{\Gamma} b$ for $(a,b,c) \in \Gamma$)

then Γ is an <u>independence reln</u> if

- invariance
- existence
- finite character $\overset{\&}{\Rightarrow}$ - monotonicity.
- transitivity
- local character
- extension
- symmetry
- Independence Thm.

more complicated in Kim-Pillay Simple Theories
this proof by Itay in JML.

<u>Theorem:</u> A cat T is simple $\Leftrightarrow$ $\exists\, \Gamma$ independence relation.

In this case, Γ = nondividing.

<u>Proof</u> ($\Rightarrow$) ✓ we have done already.

($\Leftarrow$). <u>Claim:</u> $a \underset{c}{\downarrow}^{\Gamma} b \Rightarrow a \underset{c}{\downarrow} b$

<u>Proof of Claim:</u> let ~~(a_i)~~ $(b_i : i < \kappa+1)$ be c-indiscernible in $tp(b/c)$.

Taking κ large enough (for local character ie $\kappa = |T|^+$)

let $b_\kappa = b$.

Γ-local character $\Rightarrow \exists i < \kappa$ s.t. $b \underset{cb_{<i}}{\overset{\Gamma}{\downarrow}} b_{<\kappa}$ (b_κ)

$\Rightarrow (b_j : i \leq j < \kappa)$ is Γ-Morley $/ cb_{<i}$
by Γ-invariance (& Γ-monotonicity I think).

By extension we may assume $a \underset{cb}{\overset{\Gamma}{\downarrow}} b_{<i}$.

$\Rightarrow a \underset{c}{\overset{\Gamma}{\downarrow}} b b_{<i} \Rightarrow a \underset{cb_{<i}}{\overset{\Gamma}{\downarrow}} b$

Set $p'(x, y) := tp(a, b / cb_{<i})$

$p(x, y) := tp(a, b / c)$

We want $a' \models \bigwedge_{i<j<i+\omega} p(a', b_j)$. (actually get $p' \vdash p$)

Find $(a_j \mid j < \omega)$ by extension-extraction s.t.

$(a_j b_{i+j} \mid j < \omega)$ to be indiscernible sequence $/ cb_{<i}$

in Lstp of $ab / cb_{<i}$ (let $a = a_\kappa$ then all have same Lstp as a basically).

By induction, find $(a_j' : j < \omega)$ s.t.

① $a_j' \underset{cb_{<i}}{\overset{\Gamma}{\downarrow}} b_{<i+j}$

② $a_j' \equiv^{Lstp}_{cb_{<i}} a$

③ $\forall k \lneq j \models p'(a_j', b_{i+k})$ all of these.

$a_0' := a_0$. ① $a_0 \underset{cb_{<i}}{\downarrow} b_{<i}$

② $a_0 \equiv^{Ls}_{cb_{<i}} a$ ③ ~~[illegible]~~

inductive step: Say $a_0', \dots, a_j'$ given.

So we have ① $a_j' \underset{cb_{<i}}{\perp}^{T} b_{<i+j}$ ② $a_j' \equiv^{Ls}_{cb_{<i}} a$

③ $\forall k \lneq j \models p'(a_j', b_{i+k})$.

(Going to amalgamate & use indep thm).

we also have $b_{i+j} \underset{cb_{<i}}{\perp}^{\Gamma} b_{<i+j}$ & $a_j \underset{cb_{<i}}{\perp}^{T} b_{i+j}$ and $a_j' \equiv_{c}$

by invariance

T-independence thm: $\exists a_{j+1}' \underset{cb_{<i}}{\perp}^{\Gamma} b_{<i+j+1}$ with

$a_{j+1}' \equiv^{Ls}_{cb_{<i}b_{<i+j}} a_j'$, $a_{j+1}' \equiv^{Ls}_{cb_{<i}b_{i+j}} a_j$

① ② + old ②. ③ $\models \underset{k<j+1}{p'}(a_{j+1}', b_{i}$

since for $k<j$: old ③ +
& $k=j$: & $\models p'(a_j, b_{i+k})$.

now induction is complete.

So by compactness $\exists a'$ s.t. $\models \bigwedge_{i<j<i+\omega} p'(a', b_j)$

Want to prove T is simple.

Let a be a singleton & A a set.

By T-local character, $\exists A_0 \subseteq A$ with $|A_0| < |T|^+$ s.t.

$a \underset{A_0}{\downarrow}^{\Gamma} A$.

By claim, $a \underset{A_0}{\downarrow} A$. So nondiv has local character.

So T is simple. so $(\Leftarrow)$ proved.

or the In this case:) : want $a \underset{c}{\downarrow}^{\Gamma} b \Leftrightarrow a \underset{c}{\downarrow} b$.

Pf $\Rightarrow$ above. ✓.

$\Leftarrow$ Given $a \underset{c}{\downarrow} b$. By extension/extraction & Γ-extension & Γ-local character

we get a Γ-Morley sequence $(b_i : i < \kappa)$ over c

containing b.

As $a \underset{c}{\downarrow} b$, we may assume b_i is ac-indiscernible.

Consider $tp(a/cb_{<\kappa})$.

Γ-Local char + κ big enough $\Rightarrow \exists i < \kappa$ s.t. $a \underset{cb_{<i}}{\downarrow}^{\Gamma} b_{<\kappa}$

So $a \underset{cb_{<i}}{\downarrow}^{\Gamma} b_{<\kappa} \overset{\text{monotonicity}}{\Longrightarrow} a \underset{cb_{<i}}{\downarrow}^{\Gamma} b_i \overset{\text{trans } \& \ b_i \underset{c}{\downarrow}^{\Gamma} b_{<i}}{\Longrightarrow} ab_{<i} \underset{c}{\downarrow}^{\Gamma} b_i$

$\Rightarrow a \underset{c}{\downarrow}^{\Gamma} b_i \overset{b_i \equiv_{ac} b}{\Longrightarrow} a \underset{c}{\downarrow}^{\Gamma} b$. $\square$

ack to Itay... we were proving

hm If $p(x,a)$ is an amalgamation base & let $c = a/a_E = cb(p)$.
then ① An aut fixes $c \Leftrightarrow$ fixes $p/\parallel$ setwise (cor. $p \parallel q \Rightarrow cb(p)$ & $cb(q)$ are interdefinable)
② p dnd$/c$. ③ $p|_c$ is an amalgamation base

(4) if $b \in dcl(a)$ & p dnd$/b$ then ~~[illegible]~~ $c \in bdd(b)$.

If moreover $p|_b$ is an amalgamation base then $c \in dcl(b)$.

Proof of (4) Assume p dnd$/b$ and $p|_b$ is an amalgamation base.

Then p, $p|_b$ have a common nd extn (namely p).

$\Rightarrow$ c is interdefinable with $cb(p|_b)$ ~~[illegible]~~ $= \underset{\in dcl(b)}{b/E^*}$.

$\Rightarrow$ $c \in dcl(b)$.

Since p is an amalgamation base over a, it has a unique n.d. extension to $bdd(a)$.

So we may assume $a = bdd(a)$. [at most we replace c with something interdefinable]

Since p dnd$/b$ it dnd$/bdd\ b$. (and $bdd(b) \subseteq a$).

Also $p|_{bdd(b)}$ is an amalgamation base

By the previous argument $c \in dcl(bdd(b)) = bdd(b)$. $\square$

Assumption: The formula $x \neq y$ is positive.

(eg T a first order theory without hyperimaginary sorts.)
can't pretend an infinite tuple is finite. ie only want to work with finite tuples.

Here a, b, c denote finite tuples. A, B, C = sets.
Types are in finitely many variables.

Defn For every complete type p, we define $SU(p) \in Ord \cup \{\infty\}$ as follows:

1. $SU(p) \geq 0$
2. If $SU(p) \geq \beta$ $\forall \beta \leq \alpha$ limit, then $SU(p) \geq \alpha$.
3. $SU(p) \geq \alpha + 1$ if it has a dividing extension q s.t. $SU(q) \geq \alpha$.

If $SU(p) \geq \alpha$ $\forall \alpha$, then $SU(p) = \infty$
otherwise $SU(p) = \sup\{\alpha : SU(p) \geq \alpha\}$.

Lemma TFAE:

1. $\forall p$ (in finitely many variables), $SU(p) < \infty$
2. $\forall p$ in a single variable, $SU(p) < \infty$
3. $\forall$ singleton a, & set A: $\exists A_0 \subseteq A$ finite s.t. $a \underset{A_0}{\downarrow} A$
4. $\forall$ finite a & set A: $\exists A_0 \subseteq A$ finite s.t. $a \underset{A_0}{\downarrow} A$.

<u>Proof</u> (1) $\Rightarrow$ (2) ✓.

(2) $\Rightarrow$ (3). Assume not. Then $\exists\, a, A$ s.t.

$\forall$ finite $A_0 \subseteq A$, $a \not\downarrow_{A_0} A$.

$A_0 = \emptyset$. Given $A_n \subseteq A$ finite. We know $a \not\downarrow_{A_n} A$.

$\Rightarrow \exists\, b_n \in A$ s.t. $a \downarrow_{A_n} b_n$
$\uparrow$ finite.

Define $A_{n+1} = A_n b_n$.

Then $tp(a/A_0) \subseteq a/A_1 \subseteq \dots \subseteq a/A_n \subseteq \dots$ is an infinite dividing sequence.

$\Rightarrow SU(a/\emptyset) = \infty$.

(3) $\Rightarrow$ (4). $\bar{a} = a_0 \dots a_{n+1}$.

Find $\forall i < n$ $A_i \subseteq A$ finite s.t. $a_i \downarrow_{A_i a_{<i}} A a_{<i}$

Let $B = \bigcup_{i<n} A_i$ finite. and $a_i \downarrow_{B a_{<i}} A$ $\forall i$.

By induction & transitivity $\Rightarrow \bar{a} \downarrow_B A$.

(4) $\Rightarrow$ (1). Assume $SU(p) = \infty$.

<u>Fact</u> (to be proved in a sec): If q is a nondividing ext of p then $SU(q) = SU(p)$.

Claim: $\exists \alpha$ s.t. $\forall p$ if $SU(p) \neq \infty$ then $SU(p) \leq \alpha$.

Proof of Claim: By Fact and ④, every type has the same rank as a type over a finite set. ie $SU(p) = SU(a/b)$ for some a, b finite, & the later is determined by the $tp(a, b)$ (since SU aut invariant) and there is a set of these. so take supremum. $\square$ claim

Now let α be as in the claim.

So $SU(p) = \infty \Rightarrow SU(p) \geq \alpha + 2$

$\Rightarrow$ has a dividing extension p' with $SU(p') \geq \alpha+1 \Rightarrow SU(p') = \infty$.

So we have $p = p_0 \subseteq p_1 \subseteq p_2 \subseteq \cdots$ a dividing chain.
$\in S(A_0)$ $\in S(A_1)$ $\in S(A_2)$ ...

Let $q = \bigcup p_i \in S(B)$, where $B = \bigcup A_i$.

then q divides over any finite $B_0 \subseteq B$ $\square$.

Now proof of fact:

Lemma Assume $p \subseteq q$ where $p \in S(A)$, $q \in S(B)$ $A \subseteq B$.

① $SU(q) \leq SU(p)$
② If q dnd/A then $SU(q) = SU(p)$.
③ If $SU(q) = SU(p) < \infty$ then q dnd/A.

Proof ①. Easy induction on α: $SU(q) \geq \alpha \Rightarrow SU(p) \geq \alpha$.

②. $p = tp(a/A)$, $q = tp(a/B)$.

assumption says $a \underset{A}{\downarrow} B$.

Prove ~~contrapositive~~ by induction: ie $SU(p) \geq \alpha \Rightarrow SU(q) \geq \alpha$.

$\alpha = 0$ & limit ✓.

So assume $SU(p) \geq \alpha + 1$. So $\exists c$ st. $a \underset{A}{\not\downarrow} c$ and

$SU(a/Ac) \geq \alpha$.

We may assume $c \underset{Aa}{\downarrow} B \overset{trans}{\Rightarrow} ca \underset{A}{\downarrow} B \Rightarrow a \underset{AC}{\downarrow} B$.

So by induction hyp. $SU(a/Bc) \geq \alpha$.

Also: $a \underset{B}{\not\downarrow} c$ (Otherwise $a \underset{B}{\downarrow} c \Rightarrow a \underset{A}{\downarrow} Bc \Rightarrow a \underset{A}{\downarrow} c$).

$\Rightarrow SU(q) \geq \alpha + 1$.

③ Immediate. Otherwise $SU(q) + 1 \leq SU(p)$. $\square$.

Defn If $SU(p) < \infty$ $\forall p$ then T is <u>supersimple</u>.

Fact $\forall$ ordinals α, $\exists! k < \omega$, $\alpha_0 > \alpha_1 > \dots > \alpha_{k-1}$, $\bar{n} \in \omega^k \setminus \{0\}^k$

st. $\alpha = \sum_{i<k} \omega^{\alpha_i} n_i = \omega^{\alpha_0} n_0 + \omega^{\alpha_1} n_1 + \dots + \omega^{\alpha_{k-1}} n_{k-1}$

Symmetric addition: $\sum_{i<k} \omega^{\alpha_i} n_i \oplus \sum_{i<k} \omega^{\alpha_i} m_i := \sum \omega^{\alpha_i}(n_i + m_i)$

$(n_i, m_i \in \omega)$.