We proved: $a \overset{LS}{\underset{A}{\equiv}} b$ is type-definable over A, say by $E(x,y,A)$.

for re

Define $E'(x, Z, y, W)$ as $Z = W \wedge (E(x,y,Z) \vee x = y)$.

Then E' is a type-definable equivalence relation.

Let $c = (a, A)/_{E'}$.

What does the type of a/c say? Let $p(x) := tp(a/c)$;

It says: [illegible]

"$\exists B$ s.t. $c = (x, B)/E'$."

which implies $B = A$ ie ~~that~~ $(x, B)\, E'\, (a, A)$

$\iff B = A$ & $x \overset{LS}{\underset{A}{\equiv}} a$.

In other words $tp(a/c) = Lstp(a/A)$.

You can do even better than this:

Let $bdd(A)$ = {all hyperimaginaries with small orbits over

bounded closure of A. Cheating since is a proper class but can get around

Then $c \in bdd(A)$.

But then $tp(a/c) \vdash Lstp(a/A) \vdash tp(a/bdd(A)) \vdash tp(a$

since $a \underset{bdd(A)}{\equiv} b$ is a bounded A-invariant eq re

since an automorphism fixing A ptwise fixes $bdd(A)$ setwise

So we can conclude $\mathrm{Lstp}(a/A) = tp(a/\mathrm{bdd}(A))$.

(Analogous to $stp(a/A) = tp(a/acl^{eq}(A))$)

<u>Defn</u>: A type-definable equivalence relation $E(x,y)$ is <u>small</u> if $|x|, |y| \leq |T|$ $(\Rightarrow |E| \leq |T|)$.

If E is small then the hyperimaginary sort x/E is also called small.

<u>Exercise/Remark</u>: Every type-definable eq reln $E(x,y)$ can be written as $E(x,y) \equiv \bigwedge E_i(x_i, y_i)$ where $x_i \subseteq x$, $y_i \subseteq y$ and E_i is small.

This remark implies every hyperimaginary is interdefinable with a tuple of small ones.

$a_E \rightsquigarrow (a_{i E_i}, i$ in conjunction$)$

(an aut fixes one $\Leftrightarrow$ fixes the other) ie $tp(a/b)$ & $tp(b/a)$ have unique realisations.

<u>Defn</u> $\mathcal{U}^{heq} = \mathcal{U} +$ all small h.i. sorts.

hyperimaginary

<u>Fact 2</u>: $\mathcal{U}$ is simple $\Leftrightarrow$ $\mathcal{U}^{heq}$ is.

<u>Proof</u> (Sketch) If $\mathcal{U}$ is not simple neither is $\mathcal{U}^{heq}$.

Conversely, assume $\mathcal{U}^{heq}$ is not simple.

$\Rightarrow \exists$ hi sorts x_E, y_F and formulas $\varphi(x_E, y_F)$, $\psi((y_F)_0, \ldots, (y_F)_{k-1})$ s.t. ψ is a k-inconsistency witness for φ, and $D(x_E = x_E, \Xi)$ contains arbitrarily long sequences $((\varphi,\psi), (\varphi,\psi), \ldots)$

Let $\pi(x,y) := \varphi(x/E, y/F)$
real tuples

& $\rho(y_{<k}) := \psi(y_0/F, \ldots, y_{k-1}/F)$.

These are partial types of real variables.

Also: $\rho(y_{<k}) \wedge \bigwedge_{i<k} \pi(x, y_i)$ is inconsistent.

So by compactness find $\varphi' \in \pi$ and $\psi' \in \rho$ s.t.

$\psi'(y_{<k})$ is a k-inconsistency witness for φ'.

Now prove by induction on n that

$$(\underbrace{(\varphi,\psi), \ldots, (\varphi,\psi)}_{n \text{ times}}) \in D(x_E = x_E, \Xi) \overset{*}{\Rightarrow}$$

$$(\underbrace{(\varphi',\psi'), \ldots, (\varphi',\psi')}_{n \text{ times.}}) \in D(x = x, \Xi).$$

$\Rightarrow$ * holds $\forall n \Rightarrow \mathfrak{U}$ not simple. $\square$

Generically Transitive Relations

Defn A ~~#~~ relation $R(x,y)$ is generically transitive
aRb & bRc & $a \underset{b}{\downarrow} c \Rightarrow aRc$.

(cannonial base satisfies this...)

Let us fix a type-definable (over ϕ), reflexive, symmetric, generically-transitive relation R.

Let R_n denote its n-iterate.

$a R_n b \Leftrightarrow \exists a = a_0, a_1, a_2, \ldots, a_n = b, \ a_i R a_{i+1}$.

~~lemma~~

Say that $a \tilde{R} b \Leftrightarrow aRb$ and $D(a/b, \equiv)$ contains a maximal element of $D(R(x,b), \equiv)$ $[\wedge\, tp(a)]$ ← check the rest of lecture works with this

<u>Lemma 1</u> $\forall a, b$: $a R_n b$ iff $\exists a_0 = a, a_1, a_2, \ldots, a_n = b$

s.t. $\forall i < n$ $a_i R a_{i+1}$ and $a_i \equiv a$ (ie $tp(a_i) = tp(a)$)

<u>Proof</u> $\Leftarrow$ ✓

$\Rightarrow$

$$\begin{tikzcd}
a_0 \arrow[r, dash] \arrow[dr, dash] & a_1 \arrow[d] \arrow[r, dash] & a_2 \arrow[dl, dash] & \cdots & b \\
& a_1' & & &
\end{tikzcd}$$

& $a_1' \equiv_{a_1} a_0$ & $a_1' \mathop{\downarrow}_{a_1} a_0 a_2$

(by extn)

$\Downarrow$

$a_1' R a_1$ & $a_1 R a_0$ &

$\Rightarrow a_0 R a_1'$

& similarly $a_1 R a_2 \Rightarrow a_2 R a_1'$

Now $a_1' \equiv a_0$ & proceed by induction.

<u>Lemma 2</u> Assume that $a \tilde{R} b$, bRc, and $a \mathop{\downarrow}_{b} c$, and $b \equiv c$.

Then $a \tilde{R} c$ and $a \mathop{\downarrow}_{c} b$.

Proof Since $a \tilde{R} b$, by defn there exists $\xi \in D(R(x,b), \Xi)$ maximal st. $\xi \in D(a/b, \Xi)$.

Since $b \equiv c$, $\xi \in D(R(x,c), \Xi)$ and is maximal.

Since $a \underset{b}{\downarrow} c$, $\xi \in D(a/bc, \Xi) \subseteq D(a/c, \Xi)$.

By generic transitivity, $a R c$

So $tp(a/c) \vdash R(x,c)$.

$\Rightarrow D(a/c, \Xi) \subseteq D(R(x,c), \Xi)$.

So we have ① $a \tilde{R} c$
② ξ is maximal in $D(a/c, \Xi)$.

$\Rightarrow a \underset{c}{\downarrow} b$ $\square$.

Let R^* be the transitive closure of R.

Then R^* is an equivalence relation.

Now assume $a R^* b$. Then $\exists n$ st. $a R_n b$.

Find $a_0 = a, a_1, a_2, \ldots, a_n = b$ st $\forall i < n$ $a_i R a_{i+1}$ & $a_i \equiv a$ (by lemma 1)

Find some $\xi \in D(R(x,a), \Xi)$ maximal

$\Rightarrow R(x,a) \wedge \mathrm{div}_{\xi,a}(x)$ is consistent, so find a realisation c.

Then $c \tilde{R} a$.

We may assume $c \mathop{\downarrow}_{a} a_{\leq n}$

By induction on $i < n$: $c \tilde{R} a_i$ and $c \mathop{\downarrow}_{a_i} a_{\leq n}$.

$i = 0$: $a_0 = a$ ✓.

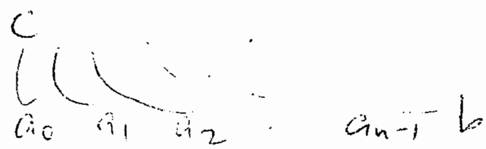
$i+1 < n$: By assumption $c \tilde{R} a_i$ & $a_i R a_{i+1}$ & $c \mathop{\downarrow}_{a_i} a_{i+1}$
& $a_i \equiv a \equiv a_{i+1}$.

$\Rightarrow c \tilde{R} a_{i+1}$ and $c \mathop{\downarrow}_{a_{i+1}} a_i$.

But $c \mathop{\downarrow}_{a_i} a_{\leq n} \Rightarrow c \mathop{\downarrow}_{a_i a_{i+1}} a_{\leq n} \Rightarrow c \mathop{\downarrow}_{a_{i+1}} a_{\leq n}$.

In particular: $a \mathop{\downarrow}_{a_{n-1}} b$, $c R a_{n-1}$, $a_{n-1} R b \Rightarrow c R b$.

$\Rightarrow R^* = R_2$.



Conclusions

① R^* is type-definable. ✓ 10U.

~~② $a \tilde{R} b \iff a R^* b$ and $a \mathop{\downarrow}_{a_{R^*}} b \iff a R b$ and $a \mathop{\downarrow}_{a_{R^*}} b$~~

Proof ②. $a \tilde{R} b \Rightarrow a R^* b$
$\xi \in D(R(x, b), \equiv)$ maximal.

should be defn of $\tilde{R}$.

<u>Defn</u> $a \tilde{R}' b \iff a R b \;\&\; \exists \xi \in D(R(x,b) \wedge tp$ maximal and $\xi \in D(a/b, \equiv)$.

<u>Defn</u> A complete type $p(x)$ over a (we may write it as $p(x,a)$) is an <u>amalgamation base</u> if the independence theorem holds for extensions of p, ie: if $q_0(x), q_1(x)$ are nondividing extensions of p to ab_0, ab_1 respectively and $b_0 \underset{a}{\perp} b_1$, then $q_0 \cup q_1$ dnd/a

(Independence thm states that Lstrong types are amalgamation bases)

~~Fix an amalgamation base $p(x,a)$.~~

should be 1-p

<u>Defn</u> Say that two amalgamation bases $p \& q$ are <u>parallel</u> if they have a common nondividing extension.

Now fix an amalgamation base $p(x,a)$ (over a).

Let $r(y) = tp(a)$.

still amalgamation bases

For $b, c \models r$, say bRc if $p(x,b)$ and $p(x,c)$ are parallel.

For other b,c: $bRc \iff b=c$.

Otherwise, $\exists d_0 \models p(x,a) \wedge p(x,b)$, $d_0 \underset{a}{\downarrow} b$ & $d_0 \underset{b}{\downarrow} a$

similarly, $\exists d_1 \models p(x,b) \wedge p(x,c)$, $d_1 \underset{b}{\downarrow} c$, & $d_1 \underset{c}{\downarrow} b$.

But all these were amalgamation bases.

ie $d_0 \underset{b}{\downarrow} a$, $d_1 \underset{b}{\downarrow} c$, $a \underset{b}{\downarrow} c$, $d_0, d_1 \models p(x,b)$

which is an amalgamation base.

$\Rightarrow \exists d \underset{b}{\downarrow} ac$, $d \underset{ab}{\equiv} d_0$, $d \underset{bc}{\equiv} d_1 \Rightarrow p(d,a), p(d,c)$.

$d \underset{b}{\downarrow} ac \Rightarrow d \underset{ab}{\downarrow} c \Rightarrow d \underset{a}{\downarrow} bc \Rightarrow d \underset{a}{\downarrow} c$

[since $d_0 \underset{a}{\downarrow} b \Rightarrow d \underset{a}{\downarrow} b$]

Similarly $d \underset{c}{\downarrow} a$. $\Rightarrow d$ realises a common nondividing extension of $p(x,a)$, $p(x,c)$ $\Rightarrow a R c$. $\square$.

3/29. After spring Break

Notation: $m D(p, \equiv) :=$ the maximal elements of $D(p, \equiv)$

$R_n =$ n-iterate of R. $E = R^*$ tr cl.

Proved: if $a E b$ then $\exists c \underset{a}{\downarrow} b$ s.t $c R a$, $c R b$.

<u>Claim:</u> If $a E b$ then $tp(a) \wedge R(x,b) \vdash tp(a/a_E)$.

<u>Proof</u>: assume $a' \models tp(a) \wedge R(x,b)$ ie $a' \equiv a$ & $a' R b$.

Then $\exists f \in Aut(\mathfrak{U})$ s.t. $f(a) = a'$.

Since $aEbRa' \Rightarrow aEa'$.

$\Rightarrow f(a_E) = a'_E = a_E$. So in fact $f \in \mathrm{Aut}(\mathfrak{U}/a_E)$

$\Rightarrow a \equiv_{a_E} a'$. $\square$

<u>Proposition</u> $a\tilde{R}b \iff aEb \wedge a \underset{a_E}{\downarrow} b$.

Moreover, this implies $D(a/a_E, \equiv) = D(tp(a) \wedge R(x,b), \equiv)$
ie RHS $\Rightarrow$
$= D(a/b, \equiv)$.

<u>Proof</u> Assume RHS: $aEb \wedge a \underset{a_E}{\downarrow} b$

Since aEb, $\exists c$ s.t. $c \underset{a}{\downarrow} b$ and $aRcRb$.

$c \underset{a}{\downarrow} b \iff c \underset{a, a_E}{\downarrow} b \Rightarrow ac \underset{a_E}{\downarrow} b \Rightarrow a \underset{c,\, a_E = c_E}{\downarrow} b \overset{\text{same ...}}{\iff} a \underset{c}{\downarrow} b$

$\boxed{\uparrow \text{in } \mathrm{dcl}(a)}$, why?

generic transitivity
$\Rightarrow aRb$.

Also $D(a/a_E, \equiv) \geq D(tp(a) \wedge R(x,b), \equiv) \underset{aRb}{=} D(a/b, \equiv)$
$\underset{\uparrow \text{independence}}{=} D(a/a_E, \equiv)$.

$\Rightarrow$ the "moreover" + $a\tilde{R}b$.

Now assume LHS: ie $a \tilde{R} b$.

~~Let~~ choose $d \underset{\substack{b_E \\ \| \\ a_E}}{\equiv} b$ st. $d \underset{a_E}{\downarrow} a$.

Then by the moreover part for a, d,

$(m)D(a/a_E, \equiv) = (m)D(tp(a) \wedge R(x,d), \equiv) \underset{\substack{\text{since} \\ d \equiv b}}{=} (m)D(tp(a) \wedge R$

$a \tilde{R} b \Rightarrow D(a/b, \equiv) \cap mD(tp(a) \wedge R(x,b)) \neq \emptyset$

$\Rightarrow D(a/b, \equiv) \cap D(a/a_E, \equiv) \neq \emptyset$

$\Rightarrow a \underset{a_E}{\downarrow} b$ $\square$.

Recall Defn: A complete type $p(x) \in S(a)$ is an <u>amalgamation base</u> if for all $b_0 \underset{a}{\downarrow} b_1$, $q_i \in S(a$ nd extns of p, we have $q_0 \cup q_1$ dnd $/a$.

* <u>Fact (Exercise)</u>: ① $\forall a, b$: $a \underset{b}{\downarrow} bdd(b)$.

② $p \in S(a)$ is an amalgamation base $\Leftrightarrow$ p has a unique ext. to $bdd(a)$ $\Leftrightarrow$ p is a Lstp.

Defn Let $p \in S(a)$, $q \in S(b)$ be amalgamation bases.

$p \parallel_1 q$ (<u>p is 1-parallel to q</u>) if they have a common nd. extn ie $\exists\, t \models p \wedge q$, $t \underset{a}{\downarrow} b$, $t \underset{b}{\downarrow} a$.

$\parallel_n$ (n-parallel) := n-iterate of $\parallel_1$

$\parallel$ (parallel) := tr cl.

<u>Eg</u> if $p \in S(a)$ is an amal base, $q \in S(a,b)$ is a nd extn of p & is an amal base then $p \parallel_1 q$.

From now (for discussion of can bases.) on all types are amalgamation bases (a.b.).

"$p(x,a)$" etc are complete types over a etc.

<u>Lemma</u> Assume $p(x,a) \parallel_1 q(x,b) \parallel_1 r(x,c)$ and $a \underset{b}{\downarrow} c$, then $p(x,a) \parallel_1 r(x,c)$.

Proof same as similar in last lecture (slightly less general). □

~~Defn~~ Fix $p(x,a)$ (actually fix $p(x,y)$ - a will vary)

<u>Defn</u> $a R b := p(x,a) \parallel_1 p(x,b)$.

So we saw that R is type-definable;
it is clearly reflexive & symmetric & by lemma generically transitive. let R_n & $E = R^*$ be as befo

<u>Definition</u> $Cb(p(x,a)) := a_E$ (the <u>cannonical b</u>ase) of $p(x,a)$.

<u>lemma</u> Assume $p \parallel_n q$ & $n>1$. Write $p = p(x,a)$.
then there exists $a' \equiv a$ s.t. $p(x,a) \parallel_1 p(x,a') \parallel$

<u>Proof</u> we have $p(x,a) \parallel_1 p_1(x,b) \parallel_1 p_2(x,c) \parallel_{n-2}$

let $a' \equiv_b a$, $a' \underset{b}{\downarrow} ac \Rightarrow p(x,a') \parallel_1 p_1(x,b)$

$a' \underset{b}{\downarrow} a \Rightarrow p(x,a) \parallel_1 p(x,a')$.

$a' \underset{b}{\downarrow} c \Rightarrow p_2(x,c) \parallel_1 p(x,a') \Rightarrow p(x,a') \parallel_{n-1} q$

<u>Cor</u> If $p(x,a) \parallel_n p(x,b)$ then $\exists a_0 = a, a_1, \ldots$
s.t. $p(x,a_i) \parallel_1 p(x,a_{i+1})$ $\forall i < n$.

<u>Cor</u> $a E b \Leftrightarrow p(x,a) \parallel p(x,b)$

<u>Proof</u> $\Rightarrow$ trivial $\Leftarrow$ by prev cor.

Theorem With previous conventions & notations:

① a_E $(= \cancel{cb(p)}$ $cb(p(x,a)))$ is a cannonical parameter for the parallelism class of $p(x,a)$ ie: $f \in Aut(\mathfrak{U})$ fixes a_E iff it fixes $P/_{\parallel}$ setwise (classwise).

② $p(x,a)$ dnd $/a_E$

③ $p(x,a)|_{a_E}$ is an amalgamation base.

④ If $b \in dcl(a)$ and $p(x,a)$ dnd$/b$ then $a_E \in bdd(b)$.

If moreover $p(x,a)|_b$ is an amalgamation base then $a_E \in dcl(b)$.

(shows a_E is minimal wrt 2&3).

→ means $p(x,a)|_b$ (where $b \in dcl(a)) := \{\varphi(x,b) : p(x,a) \vdash \varphi(x,b)\}$
$= tp(t/b) \ \forall t \models p(x,a)$.

Proof ① Assume f fixes a_E. Then $a \, E \, f(a) \underset{\text{by con.}}{\Leftrightarrow}$
$p(x,a) \parallel p(x,f(a)) \underset{\text{always.}}{\Leftrightarrow} f(P/_{\parallel}) = P/_{\parallel}$ setwise.

② Choose $b \equiv_{a_E} a$ s.t. $b \underset{a_E}{\downarrow} a$. Then $b E a$
$\Rightarrow a \tilde{R} b \Rightarrow p(x,a) \parallel_1 p(x,b) \Rightarrow \exists t$ s.t. $t \models p(x,a) \wedge p(x,b)$ &

$t \underset{a}{\not\downarrow} b$ & $t \underset{b}{\downarrow} a$.

Recall: $t \underset{b}{\downarrow} a \Rightarrow t \underset{b,a_E}{\downarrow} a \Rightarrow tb \underset{a_E}{\downarrow} a \Rightarrow t \underset{a_E}{\downarrow}$

$\Rightarrow p(x,a)$ dnd $/a_E$.

3. Assume we have $b_0 \underset{a_E}{\downarrow} b_1$, $t_i \underset{a_E}{\downarrow} b_i$, $t_i \models p(x;$

We need to find $t \underset{a_E}{\downarrow} b_0 b_1$ s.t. $t \underset{a_E b_i}{\equiv} t_i$.

Since everything happens over a_E, we may assume

$a \underset{a_E}{\downarrow} b_0 b_1 (t_0 t_1)$.

$\Rightarrow a \underset{a_E b_0}{\downarrow} b_1 \Rightarrow a b_0 \underset{a_E}{\downarrow} b_1 \Rightarrow b_0 \underset{a}{\downarrow} b_1$

We know $t_0 \models p(x,a) |_{a_E}$

$\Rightarrow \exists a_0 \underset{a_E}{\equiv} a$ s.t. $t_0 \models p(x, a_0)$.

$\Rightarrow \exists s \models p(x,a)$ then $s \equiv_{a_E} t_0$ so

find a_0 st. $s, a \underset{a_E}{\equiv} t_0 a_0$. $\Rightarrow$

~~We may assume $tp(t_0/b_0 a_E)$ is ...~~

Want to prove ind thm for $a \underset{a_E}{\downarrow} b_0$ $t_0 \underset{a_E}{\downarrow} b_0$ $s \underset{a_E}{\downarrow}$

$\exists a_0$ st. $a_0 E a$ and $p(t_0, a_0)$. wma $a_0 \underset{a_E t_0}{\downarrow} a b_0$

$\Rightarrow a_0 \underset{a_E}{\downarrow} t_0 b_0 \Rightarrow t_0 \underset{b_0 a_E}{\downarrow} a_0 \Rightarrow a_0 t_0 \underset{a_E}{\downarrow} b_0 \Rightarrow$

WMA that $b_0 = \mathrm{bdd}(b_0, a_E)$. (Check assumptions still hold).

$$\Rightarrow \boxed{\begin{array}{l} tp(t_0/a_0) \parallel_1 tp(t_0/b_0). \\ \;\| \\ p(x,a_0). \end{array}}$$

Since $a_0 \downarrow_{a_E} a$ and $a_0 E a$: $\boxed{p(x,a) \parallel_1 p(x,a_0)}$.

& finally $a_0 \downarrow_{a_E t_0} a b_0 \Rightarrow a_0 \downarrow_{a_E} a b_0 t_0$

$\Rightarrow a_0 \downarrow_{a, a_E} b_0 \Rightarrow a_0 a \downarrow_{a_E} b_0 \Rightarrow \boxed{a \downarrow_{a_0} b_0}$.

$\square + \square + \square \Rightarrow p(x,a) \parallel_1 tp(t_0/b_0)$.

$\Rightarrow \exists\, t_0'$ s.t. $t_0' \downarrow_{a} b_0$; $t_0' \downarrow_{b_0} a$

$$\Updownarrow$$

$t_0' \downarrow_{a_E} b_0 a$ since $t_0' \equiv_{a_0 b_0} t_0$ & $t_0' \models p(x,a)$.

Need $\square + \square$

Similarly find $t_1' \downarrow_a b_1$ st. $t_1' \models p(x,a)$, $t_1' \equiv_{a_E b_1} t_1$

So we have $b_0 \downarrow_a b_1$, $t_0' \downarrow_a b_0$, $t_1' \downarrow_a b$

$t_1', t_0' \models p(x,a) \Rightarrow \exists\, t \downarrow_a b_0 b_1$

$\left(\Rightarrow t \downarrow_{a_E} b_0 b_1 a\right)$

s.t. $t \equiv_{\alpha\beta_i} t_i' \equiv_{\alpha_E\beta\kappa} t_i$ $\square$.