18.969 Topics in Geometry: Mirror Symmetry
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MIRROR SYMMETRY: LECTURE 18

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1. Derived Fukaya Category

Last time: derived categories for abelian categories (e.g. $D^b\mathrm{Coh}(X)$). This time: the derived Fukaya category. We start with an A_∞-category $\mathcal{A}$ and obtain a triangulated category via "twisted complexes". Recall that in an A_∞-category, $\hom_{\mathcal{A}}(X,Y)$ is a graded vector space equipped with maps

$$m_k : \hom_{\mathcal{A}}(X_0,X_1)\otimes\cdots\otimes\hom_{\mathcal{A}}(X_{k-1},X_k)\to\hom_{\mathcal{A}}(X_0,X_k)[2-k] \tag{1}$$

1) Additive enlargement: we define the category ΣA to be the category whose objects are finite sums $\bigoplus X_i[k_i], X_i\in\mathcal{A}, k_i\in\mathbb{Z}$ and whose maps are

$$\hom_{\Sigma\mathcal{A}}(\bigoplus_i X_i[k_i],\bigoplus_j Y_j[\ell_j])=\bigoplus_{i,j}\hom_{\mathcal{A}}(X_i,Y_j)[\ell_j-k_i] \tag{2}$$

Note that we have induced multiplication maps

$$m_k(a_k,\ldots,a_1)^{ij}=\sum_{i_1,\ldots,i_{k-1}} m_k(a_k^{i_{k-1},j},\ldots,a_1^{i_1,j}) \tag{3}$$

2) Twisted complexes: we define the category $\mathrm{Tw}\mathcal{A}$ to be the category whose objects are twisted complexes (X,δ_X),

$$X=\bigoplus_i X_i[k_i]\in\Sigma\mathcal{A}, \delta_X=(\delta_X^{ij})\in\hom^1_{\Sigma\mathcal{A}}(X,X) \tag{4}$$

(i.e. δ_X a degree 1 endomorphism) s.t.

- δ_X is strictly lower-triangular, and
- $\sum_{k=1}^\infty m_k(\delta_x,\ldots,\delta_x)=0$. It is a finite sum because δ_X is lower triangular, and generalizes $\delta_X\circ\delta_X=0$.

Example. For a simple map $f:X_1\to X_2, f\in\hom^1_{\mathcal{A}}(X_1,X_2)$, the condition is $m_1(f)=0$. Now, for maps $X_1[2]\xrightarrow{f}X_2[1]\xrightarrow{g}X_3$ and $X_1[2]\xrightarrow{h}X_3$,

$$\begin{aligned}
g&\in\hom^0(X_2,X_3)=\hom^1(X_2[1],X_3)\\
f&\in\hom^0(X_1[1],X_2[1])=\hom^1(X_1[2],X_2[1])\\
h&\in\hom^{-1}(X_1,X_3)=\hom^1(X_1[2],X_3)
\end{aligned} \tag{5}$$

the condition is $m_1(f)=m_1(g)=0$ and $m_2(g,f)+m_1(h)=0$.

The morphisms in the category of twisted complexes are

$$\hom_{\mathrm{Tw}\mathcal{A}}((X,\delta_X),(Y,\delta_Y)) = \hom_{\Sigma\mathcal{A}}(X,Y) \tag{6}$$

and

$$\begin{aligned} m_k^{\mathrm{Tw}\mathcal{A}}(a_k,\dots,a_1) = \sum_{i_0,\dots,i_k} \pm m^{\Sigma\mathcal{A}}_{k+i_0+\dots+i_k}(&\underbrace{\delta_{X_k},\dots,\delta_{X_k}}_{i_k}, a_k, \underbrace{\delta_{X_{k-1}},\dots,\delta_{X_{k-1}}}_{i_{k-1}}, \\ &\dots, \underbrace{\delta_{X_1},\dots,\delta_{X_1}}_{i_1}, a_1, \underbrace{\delta_{X_0},\dots,\delta_{X_0}}_{i_0}) \end{aligned} \tag{7}$$

Tw$\mathcal{A}$ is a *triangulated* A_∞-category, i.e. there are mapping cones satisfying the usual axioms.

Example. For $a \in \hom(X,Y)$,

$$m_1^{\mathrm{Tw}\mathcal{A}}(a) = m_1(a) \pm m_2(\delta_Y, a) \pm m_2(a,\delta_X) + \text{higher terms} \tag{8}$$

This is a generalization of being a chain map up to homotopy.

3) We now take the cohomology category $D(\mathcal{A}) := H^0(\mathrm{Tw}\mathcal{A})$, which is an honest triangulated category. The objects of the two categories are the same, but now our morphisms are $\hom^{D(\mathcal{A})}(X,Y) := H^0(\hom^{\mathrm{Tw}\mathcal{A}}(X,Y), m_1^{\mathrm{Tw}(\mathcal{A})})$. Note that $\hom^{D(\mathcal{A})}(X,Y[k]) = H^k(\hom^{\mathrm{Tw}\mathcal{A}}(X,Y), m_1^{\mathrm{Tw}\mathcal{A}})$. The composition is induced by $m_2^{\mathrm{Tw}\mathcal{A}}$ on cohomology.

Remark. There is a variant of this called a *split-closed derived category.* Let $\mathcal{A}$ be a linear category, $X \in \mathcal{A}, p \in \hom_{\mathcal{A}}(X,X)$ s.t. $p^2 = p$ (idempotent). Define the image of p to be an object Y, and add maps $u : X \to Y, v : Y \to X$ s.t. $u \circ v = \mathrm{id}_Y, v \circ u = p$. That is, we enlarge $\mathcal{A}$ to add these objects and maps, and define the split closure to be the category whose objects are (X,p) with p idempotent, and morphisms $\hom((X,p),(Y,p')) = p' \hom(X,Y)p$. This is more complicated in the A_∞ setting.

Geometrically, some exact triangles in $D\mathrm{Fuk}(M)$ are given by Lagrangian connected sums (FOOO) and Dehn twists (Seidel).

- For an example of the latter, given a cylinder with a Lagrangian circle S, we can obtain a symplectomorphism $\tau_S \in \mathrm{Symp}(M,\omega)$ which is the identity outside a neighborhood of S and, within that neighborhood, twists the cylinder around (in higher dimensions, define this using the geodesic flow in a neighborhood of $S \cong T^*S$). If L is Lagrangian, then $\tau_S(L)$ is Lagrangian as well. By Seidel, there exists an exact triangle in

$D\mathrm{Fuk}(M)$:

(9)
$$\begin{tikzcd} HF^*(S,L)\otimes S \arrow[rr, "t"] & & L \arrow[dl] \\ & \tau_S(L) \arrow[ul, "{[1]}"] & \end{tikzcd}$$

These correspond to long exact sequences for $HF(L', -)$.

- In the former situation, for L_1, L_2 (graded) Lagrangians, $L_1 \cap L_2 = \{p\}$ of index 0, we can construct the connected sum $L_1 \#_p L_2$, which looks locally like $\tau_{L_1}(L_2)$ if L_1 is a sphere and is given by $\mathrm{Cone}(L_1 \xrightarrow{p} L_2)$ in general (consider this vs. "$L_1[1] \cup_p L_2 \simeq \mathrm{Cone}(L_1 \xrightarrow{0} L_2)$"). For instance, in the torus T^2, consider two independent loops α of degree 2 and β of degree 1, with two points of intersection p, q. Then $\mathrm{Cone}(\alpha \xrightarrow{p+q} \beta) \simeq \gamma_1 \oplus \gamma_2$ is disconnected, where γ_1, γ_2 are degree 1 loops. If we only started with α, β, the triangulated envelope contains $\gamma_1 \oplus \gamma_2$, but not γ_1, γ_2 separately. The split-closure does contain them.
- Now, if we start with two independent generators of the torus, successive Dehn twists give all the homotopy classes of loops in T^2, but each homotopy class contains infinitely many non-Hamiltonian isotopic Lagrangians. To generate $D\mathrm{Fuk}(T^2)$ as a triangulated envelope, we need (for instance) one horizontal loop and infinitely many vertical loops. On the other hand, α, β above are split generators. The key point is that $\mathrm{Cone}(\alpha \xrightarrow{p+T^q q} \beta)$ gives deformed loops, direct sums of which vary continuously within a homotopy class. But many cones and idempotents have no obvious geometric interpretation. For instance, the Clifford torus $T = \{|x| = |y| = |z|\} \subset \mathbb{CP}^2$ has idempotents in $HF(T, T)$ without any obvious geometric interpretation.