18.969 Topics in Geometry, MIT Fall term, 2006

Problem sheet 2

Exercise 1. Let $J \in \mathrm{End}(T)$ be an almost complex structure, and let $N_J = [J, J]$ be its Nijenhuis tensor, defined alternatively by

$$[J, J](X, Y) = [X, Y] - [JX, JY] + J([JX, Y] + [X, JY]).$$

Defining $\partial : \Omega^{p,q}(M) \longrightarrow \Omega^{p+1,q}(M)$ by $\partial = \pi_{p+1,q}d$ and $\overline{\partial}$ its complex conjugate, then

$$d = \partial + \overline{\partial} + d_N;$$

determine the operator d_N and its decomposition into (p, q) types.

Exercise 2. Show that S^{4k} has no almost complex structure.

Exercise 3. Let (M, J) be a complex manifold. Show that the partial connection

$$\overline{\partial}_X Y = \pi_{1,0}[X, Y], \quad X \in C^\infty(T_{0,1}),\ Y \in C^\infty(T_{1,0}),$$

defines a holomorphic structure on $T_{1,0} \cong (T, J)$, therefore called the holomorphic tangent bundle.

Show furthermore that this may also be expressed in terms of the Nijenhuis bracket

$$[J, \cdot] : \Omega^0(T) \longrightarrow \Omega^0(T).$$

Exercise 4. We saw that the space of Dirac structures $Dir(V)$ in $V \oplus V^*$ is equivalent to $O(n, \mathbb{R})$, the real orthogonal group. By choosing a generalized metric $g + b$, determine explicitly the map

$$O(V, g) \longrightarrow Dir(V)$$

sending $A \in O(V, g)$ to D_A.

Exercise 5. Let $\mathcal{D}V = V \oplus V^*$ and $\mathcal{D}W = W \oplus W^*$. Show that the graph Γ_Q of any orthogonal isomorphism $Q : \mathcal{D}V \longrightarrow \mathcal{D}W$ is a Dirac structure in $\mathcal{D}V \times \overline{\mathcal{D}W}$, where $\overline{\mathcal{D}W}$ denotes $W \oplus W^*$ with the opposite (negative) bilinear form.

Show that even if Dirac structures $\Gamma_1 \subset \mathcal{D}U \times \overline{\mathcal{D}V}$ and $\Gamma_2 \subset \mathcal{D}V \times \overline{\mathcal{D}W}$ are not graphs of orthogonal maps, they can be composed to produce a Dirac structure $\Gamma_1 \circ \Gamma_2 \subset \mathcal{D}U \times \overline{\mathcal{D}W}$.

Exercise 6. Let $C_+ \subset V \oplus V^*$ be a maximal positive-definite subspace, which must therefore be the graph of $g + b$ for $g \in S^2V^*$ and $b \in \wedge^2 V^*$. Let $G = 1|_{C_+} - 1|_{C_-}$ be the associated generalized metric, so that $\langle G\cdot, \cdot\rangle$ defines a positive-definite metric on $V \oplus V^*$.

- We saw that the restriction of G to $V \subset V \oplus V^*$ was

$$g^b = g - bg^{-1}b.$$

Show explicitly that g^b is indeed positive-definite. Also, show that its volume form is given by

$$vol_{g^b} = \det(g - bg^{-1}b)^{1/2} = \det(g+b)\det g^{-1/2}.$$

(Hint: $g - bg^{-1}b = (g-b)g^{-1}(g+b)$.)

- Let (e_i) be an oriented g-orthonormal basis for V. Show that $(a_i = e_i + (g+b)(e_i))$ form an oriented orthonormal basis for C_+. Hence $* = a_1 \cdots a_n$ is a generalized Hodge star. Show that $* \in \mathrm{Pin}(V \oplus V^*)$ covers $-G \in O(V \oplus V^*)$.

- Show explicitly that the Mukai pairing $(*1, 1) = \det(g+b)\det g^{-1/2} = vol_{g^b}$.

- Show that $vol_{g^b}/vol_g = ||e^b||_g^2$ (Hint: determine the relationship between $*_g$ and $*_{g+b}$.)

Exercise 7. Show that the derived bracket expression $[a,b]_H \cdot \varphi = [[d_H, a], b] \cdot \varphi$ for the twisted Courant bracket (where $d_H = d + H \wedge \cdot$) agrees with that obtained from the axioms of an exact Courant algebroid, i.e.

$$[X+\xi, Y+\eta]_H = [X,Y] + L_X\eta - i_Y d\xi + i_Y i_X H.$$

Exercise 8. Let $[\cdot,\cdot]$ be the derived bracket on $C^\infty(T \oplus T^*)$ of the operator $d_H = d + H \wedge \cdot$ but do not assume that $dH = 0$. Prove that

$$[[a,b],c] = [a,[b,c]] - [b,[a,c]] + i_{\pi c} i_{\pi b} i_{\pi a} dH.$$

Exercise 9. Let $\pi : T^* \longrightarrow T$ be a Poisson structure with associated Poisson bracket $\{,\}$. Show that T^* inherits a natural Lie algebroid structure, where π is the anchor map and

$$[df, dg] = d\{f,g\}.$$