Lecture 13.

14 Fiber bundles

The notion of a vector bundle has a natural and useful generalization, that of a fiber bundle. Here is a basic example.

Example 14.1. A k-frame for $\mathbb{R}^n$ is a k-tuple $(e_1, \ldots, e_k)$ of linearly independent vectors.

Let $\mathrm{St}_k(\mathbb{R}^n)$ be the space of all k-frames for $\mathbb{R}^n$. This the Stiefel manifold. There is a natural map

$$p\colon \mathrm{St}_k(\mathbb{R}^n) \to \mathrm{Gr}_k(\mathbb{R}^n)$$

given by sending the k-tuple to $(v_1, v_2, \ldots, v_k)$ to its span. This map is a submersion and the preimage of small open sets can be given a product structure.

Definition 14.2. A (locally trivial) fiber bundle with fiber F is triple (E, B, p) where $p\colon E \to B$ is a smooth map so that for all $b \in B$ in B there is a neighborhood U of b and a diffeomorphism:

$$\tau\colon p^{-1}(U) \to U \times F$$

so that $p_1 \circ \tau = p$ where $p_1\colon U \times F \to U$ is the projection.

In our example let U_Π be one of our standard charts and let $F = \mathrm{Inj}(\mathbb{R}^k, \mathbb{R}^n)$ be the space of injective linear maps. This an open subset of $\hom(\mathbb{R}^k, \mathbb{R}^n)$ so it is a manifold. We'll define the inverse of the trivialization

$$\tau^{-1} : U_\Pi \times F \to p^{-1}(U_\Pi).$$

To do this we need to fix an identification of $\iota\colon \Pi \to \mathbb{R}^k$. Then

$$\tau^{-1}(\Gamma_A, j) = (A \circ \iota \circ je_1), A \circ \iota \circ j(e_2), \ldots, A \circ \iota \circ j(e_k)).$$

where as usual $A\colon \Pi \to \Pi^\perp$ is a linear transformation and Γ_A is its graph.

For another example consider a real vector bundle $p\colon E \to B$. The projectivization of E, denoted $\mathbb{P}(E)$ is space of lines in E and has natural projection $p'\colon \mathbb{P}(E) \to B$ which is a fiber bundle with fiber $\mathbb{RP}^{n-1}$.