18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

The Adem Relations (Continued) (Lecture 5)

We continue to work with complexes over the finite field $\mathbf{F}_2$ with two elements. All homology and cohomology will be taken with coefficients in $\mathbf{F}_2$.

In the last lecture, we showed how to reduce the proof of the Adem relations to a calculation in group homology. Our goal in this lecture is to carry out that calculation. We begin with some generalities.

Let V be a complex with an action of the group Σ_2. In previous lectures, we have made extensive use of the homotopy coinvariants construction

$$V \mapsto V_{h\Sigma_2} \simeq (V \otimes E\Sigma_2)_{\Sigma_2}.$$

There is also a dual *homotopy invariants* construction, given by

$$V \mapsto V^{h\Sigma_2} \simeq \mathrm{Hom}(E\Sigma_2, V)^{\Sigma_2}.$$

These constructions are related by a *norm map* $N : V_{h\Sigma_2} \to V^{h\Sigma_2}$, which has the property that the composition

$$V \to V_{h\Sigma_2} \xrightarrow{N} V^{h\Sigma_2} \to V$$

coincides with the usual norm map $v \mapsto \sum_{g \in \Sigma_2} g(v)$. The *Tate construction* on V is defined to be the cofiber of the norm map, and will be denoted by $V^{T\Sigma_2}$. By construction, we have a fiber sequence

$$V_{h\Sigma_2} \to V^{h\Sigma_2} \to V^{T\Sigma_2},$$

which induces a long exact sequence on cohomology.

To get a feel for how everything works, let's consider the case where $V = \mathbf{F}_2$ is a complex concentrated in degree 0. In this case, we can identify $V_{h\Sigma_2}$ with the chain complex $C_*(B\Sigma_2)$, and we can identify $V^{h\Sigma_2}$ with the cochain complex $C^*(B\Sigma_2)$. The norm map induces a map

$$\mathrm{H}_n(B\Sigma_2) \to \mathrm{H}^{-n}(B\Sigma_2).$$

This is just the usual norm map in the theory of group cohomology. It vanishes for $n \neq 0$ simply for degree reasons. For $n = 0$, it is given by multiplication by the order of the group $B\Sigma_2$, and therefore vanishes because we are taking coefficients in the field $\mathbf{F}_2$. Because the norm map vanishes in this case, it is convenient to rewrite the above fiber sequence as

$$V^{h\Sigma_2} \to V^{T\Sigma_2} \to V_{h\Sigma_2}[1].$$

The cohomology of $V^{T\Sigma_2}$ is the *Tate cohomology* of the group Σ_2. The long exact sequence above gives isomorphisms

$$\mathrm{H}^n(V^{T\Sigma_2}) \simeq \mathrm{H}^n(B\Sigma_2)$$
$$\mathrm{H}^{-n-1}(V^{T\Sigma_2}) \simeq \mathrm{H}_n(B\Sigma_2)$$

for $n \geq 0$. In particular, we see that the Tate cohomology of Σ_2 is 1-dimensional in every degree.

Recall that the cohomology ring $\mathrm{H}^*(B\Sigma_2)$ is isomorphic to the polynomial ring $\mathbf{F}_2[t]$. The multiplication on $\mathrm{H}^*(B\Sigma_2)$ extends to a multiplication defined on the Tate cohomology $\mathrm{H}^*(V^{T\Sigma_2})$, which can be identified with the ring of Laurent polynomials $\mathbf{F}_2[t, t^{-1}]$. This induces an isomorphism

$$\mathrm{H}_*(B\Sigma_2) \simeq \mathbf{F}_2[t, t^{-1}]/\mathbf{F}_2[t].$$

Using this isomorphism, $\mathrm{H}_*(B\Sigma_2)$ has a basis consisting of $\{t^n\}_{n<0}$. In previous lectures, we used a basis $\{x_i\}_{i\geq 0}$ for $\mathrm{H}_*(B\Sigma_2)$ which was dual to the basis $\{t^i\}_{i\geq 0}$ for $\mathrm{H}^*(B\Sigma_2)$. By comparing degrees, we see that these bases are related by the following transformation

$$x_i \mapsto t^{-i-1}.$$

It follows that the duality pairing between homology and cohomology can be written in the following suggestive form:

$$(f, g) \mapsto \mathrm{Res}(fg).$$

Here $\mathrm{Res} : \mathbf{F}_2[t, t^{-1}] \to \mathbf{F}_2$ denotes the *residue map*, which simply extracts the coefficient of t^{-1}.

Let us now consider some more interesting Σ_2-actions. For every complex V, there is a canonical action of Σ_2 on the tensor square $V \otimes V$. We have defined the symmetric square $D_2(V)$ to be the homotopy coinvariants $(V \otimes V)_{h\Sigma_2}$. This construction has the following counterparts for homotopy invariants and the Tate construction:

$$D^2(V) = (V \otimes V)^{h\Sigma_2}$$
$$D^T(V) = (V \otimes V)^{T\Sigma_2}.$$

We now wish to describe the effects that these constructions have on cohomology. We can produce operations by repeating some of our earlier constructions.

Definition 1. Let V be a complex, and let $v \in \mathrm{H}^n(V)$, so that v classifies a map $\mathbf{F}_2[-n] \to V$. We obtain induced maps

$$f : D^2(\mathbf{F}_2)[-2n] \simeq D^2(\mathbf{F}_2[-n]) \to D^2(V)$$
$$f' : D^T(\mathbf{F}_2)[-2n] \simeq D^2(\mathbf{F}_2[-n]) \to D^T(V).$$

For every integer k, we let $S^k(v) \in \mathrm{H}^{n+k}(D^T(V))$ denote the image of $t^{k-n} \in \mathrm{H}^{k-n}(D^T(\mathbf{F}_2))$ under the map f'. If $k \geq n$, then

$$t^{k-n} \in \mathrm{H}^{k-n}(D^2(\mathbf{F}_2)) \subseteq \mathrm{H}^{k-n}(D^T(\mathbf{F}_2)).$$

In this case, we will denote the image of t^{k-n} under f by $S^k(v) \in \mathrm{H}^{n+k}(D^2(V))$.

Remark 2. Our notation is potentially ambiguous, but will hopefully not result in any confusion since for $k \geq n$, the diagram

$$\begin{array}{ccc} \mathrm{H}^n(V) & \xrightarrow{S^k} & \mathrm{H}^{n+k}(D^2(V)) \\ \downarrow = & & \downarrow \\ \mathrm{H}^n(V) & \xrightarrow{S^k} & \mathrm{H}^{n+k}(D^T(V)) \end{array}$$

is commutative.

Now suppose that V is equipped with a symmetric multiplication $m : D_2(V) \to V$. We can regard m as a homotopy fixed point for the action of Σ_2 on the space $\mathrm{Hom}(V \otimes V, V)$. Consequently, m gives rise to a commutative diagram

$$\begin{array}{ccc} D^2(V) & \xrightarrow{f'} & V^{h\Sigma_2} \\ \downarrow & & \downarrow \\ D^T(V) & \xrightarrow{f''} & V^{T\Sigma_2} \\ \downarrow & & \downarrow \\ D_2(V)[1] & \longrightarrow & V_{h\Sigma_2}[1]. \end{array}$$

Here we regard Σ_2 as acting trivially on V.

We wish to describe the induced maps on cohomology in terms the Steenrod operations on $\mathrm{H}^*(V)$. For this, we need to introduce a mild finiteness restriction on V:

($*$) The cohomology groups $\mathrm{H}^n(V)$ are finite dimensional for every $n \in \mathbf{Z}$, and vanish for n sufficiently small.

Assuming condition ($*$), we have equivalences

$$V^{h\Sigma_2} \simeq V \otimes (\mathbf{F}_2)^{h\Sigma_2}$$

$$V^{T\Sigma_2} \simeq V \otimes (\mathbf{F}_2)^{T\Sigma_2}$$

$$V_{h\Sigma_2} \simeq V \otimes (\mathbf{F}_2)_{h\Sigma_2}.$$

Passing to cohomology, we obtain isomorphisms

$$\mathrm{H}^*(V^{h\Sigma_2}) \simeq \mathrm{H}^*(V)[t]$$

$$\mathrm{H}^*(V^{T\Sigma_2}) \simeq \mathrm{H}^*(V)[t, t^{-1}]$$

$$\mathrm{H}^*(V_{h\Sigma_2}) \simeq \mathrm{H}^{*+1}(V)[t, t^{-1}] / \mathrm{H}^*(V)[t].$$

We now have the following result:

Proposition 3. *Let V be a complex equipped with a symmetric multiplication, and let $v \in \mathrm{H}^n(V)$. Then:*

(1) *If $k \geq n$, then $S^k(v) \in \mathrm{H}^{n+k}(D^2(V))$ has image*

$$\sum_l \mathrm{Sq}^l(v) t^{k-l} \in \mathrm{H}^*(V)[t].$$

(2) *For all integers k, the element $S^k(v) \in \mathrm{H}^{n+k}(D^T(V))$ has image*

$$\sum_l \mathrm{Sq}^l(v) t^{k-l} \in \mathrm{H}^*(V)[t, t^{-1}].$$

Proof. The implication (2) $\Rightarrow$ (1) is clear. To prove (2), we consider the map $\phi : \mathrm{H}^*(D^T(V)) \to \mathrm{H}^*(V)[t, t^{-1}]$. We observe that ϕ is a map of modules over the Tate cohomology ring $\mathrm{H}^*(\mathbf{F}_2^{T\Sigma_2}) \simeq \mathbf{F}_2[t, t^{-1}]$, and that the action of this ring on $\mathrm{H}^*(D^T(V))$ satisfies $t^m S^k(v) = S^{m+k}(v)$.

The coefficient of t^{k-l} in $\phi(S^k(v))$ is given by

$$\mathrm{Res}(t^{l-k-1}\phi(S^k(v))) = \mathrm{Res}(\phi(S^{l-1}(v))).$$

We have a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{H}^*(V) & \xrightarrow{S^{l-1}} & \mathrm{H}^*(D^T(V)) & \longrightarrow & \mathrm{H}^*(V)[t,t^{-1}] & & \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & \searrow^{\mathrm{Res}} & \\
\mathrm{H}^*(V) & \xrightarrow{\mathrm{Sq}^l} & \mathrm{H}^*(D_2(V)) & \longrightarrow & \mathrm{H}^*(V)[t,t^{-1}]/\,\mathrm{H}^*(V)[t] & \xrightarrow{\mathrm{Res}} & \mathrm{H}^*(V).
\end{array}$$

We now observe that the composition of the bottom arrows is the definition of the map Sq^l. □

We now wish to restrict further to the case where $V \simeq C^*(\mathbf{R}P^\infty)$ is the cochain complex which computes the cohomology of $B\Sigma_2 \simeq \mathbf{R}P^\infty$. To avoid confusion, let us identify this cohomology ring with the polynomial algebra $\mathbf{F}_2[u]$. We saw in a previous lecture that the action of the Steenrod algebra on $\mathbf{F}_2[u]$ was given by

$$\mathrm{Sq}^k(u^n) = (n-k,k)u^{n+k}.$$

Let G denote the wreath product $(\Sigma_2 \times \Sigma_2) \rtimes \Sigma_2$, so the cochain complex $C^*(BG)$ is equivalent to $D^2(C^*(\Sigma_2))$. We may view f as a map

$$C^*(BG) \to C^*(\Sigma_2)^{h\Sigma_2} \simeq C^*(\Sigma_2 \times \Sigma_2).$$

At the level of cohomology, this is simply the map induced by the inclusion of groups

$$\Sigma_2 \times \Sigma_2 \simeq \Sigma_2 \rtimes \Sigma_2 \xrightarrow{j} (\Sigma_2 \times \Sigma_2) \rtimes \Sigma_2 = G.$$

Applying Proposition 3 in this case, we obtain the following:

Corollary 4. *The inclusion $j : \Sigma_2 \times \Sigma_2 \to G$ induces a restriction map on cohomology $\mathrm{H}^*(BG) \to \mathrm{H}^*(\Sigma_2 \times \Sigma_2) \simeq \mathbf{F}_2[t,u]$. For $k \geq n$, this map carries $S^k(u^n) \in \mathrm{H}^{m+k}(BG)$ to*

$$\sum_p (n-l,l)u^{n+l}t^{k-l}.$$

We observe that $\mathrm{H}_*(BG) \simeq \mathrm{H}^{-*}(D_2(C_*(B\Sigma_2)))$ has a basis consisting of products $\{x_i x_j\}_{0 \leq i < j}$ and Steenrod operations $\{\overline{\mathrm{Sq}}^{-n}\, x_i\}_{0 \leq i \leq n}$. We obtain a dual basis for $\mathrm{H}^*(BG)$ consisting of vectors $\{v_{ij}\}_{0 \leq i < j}$ and Steenrod operations $\{S^n u^i\}_{0 \leq i \leq n}$. The basis vectors v_{ij} span the image of the norm map

$$\mathrm{H}^*(D_2(C^*(\Sigma_2))) \to \mathrm{H}^*(D^2(C^*(\Sigma_2))),$$

so the restriction map $\mathrm{H}^*(BG) \to \mathrm{H}^*(\Sigma_2 \times \Sigma_2)$ vanishes on them. Thus Corollary 4 really gives a complete description of the restriction map $\mathrm{H}^*(BG) \to \mathrm{H}^*(\Sigma_2 \times \Sigma_2)$. Rewriting this information in terms of the dual bases, we obtain the following result:

Corollary 5. *The inclusion $j : \Sigma_2 \times \Sigma_2 \to G$ induces a map on homology*

$$\mathrm{H}_*(\Sigma_2 \times \Sigma_2) \to \mathrm{H}_*(G)$$

which is described by the formula

$$x_p \otimes x_q \mapsto \sum_l (p-2l,l)\,\overline{\mathrm{Sq}}^{-q-l}\, x_{p-l}.$$

We are now ready to complete the calculation of the last lecture. Recall that we need to show that for $p, q > 0$, the homology classes

$$\sum_l (p-2l,l)\,\overline{\mathrm{Sq}}^{-q-l}\, x_{p-l} \in \mathrm{H}_{p+q}(BG)$$

$$\sum_{l'} (q-2l', l')\, \overline{\mathrm{Sq}}^{-p-l'}\, x_{q-l'} \in \mathrm{H}_{p+q}(BG)$$

have the same image in $\mathrm{H}_*(B\Sigma_4)$. Invoking Corollary 5, we see that it suffices to show that under the induced inclusion

$$\Sigma_2 \times \Sigma_2 \to \Sigma_4,$$

the homology classes $x_p \otimes x_q, x_q \otimes x_p \in \mathrm{H}_{p+q}(B(\Sigma_2 \times \Sigma_2))$ have the same image in $\mathrm{H}_{p+q}(B\Sigma_4)$. These two homology classes conjugate by the involution which permutes the two factors in the product $\Sigma_2 \times \Sigma_2$. We now observe that this involution is the restriction of an *inner* automorphism of Σ_4, and that inner automorphisms of a group H act trivially on the homology $\mathrm{H}_*(BH)$.