18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

p-adic Completion of Spaces (Lecture 31)

In this lecture, we will discuss the relationship between the category $\mathfrak{S}_p^\vee$ of p-profinite spaces and the usual category $\mathfrak{S}$ of spaces. As we have seen earlier, there is a pair of adjoint functors

$$\mathfrak{S} \underset{\varprojlim}{\overset{\vee}{\rightleftarrows}} \mathfrak{S}_p^\vee .$$

The composition

$$X \mapsto \varprojlim X^\vee$$

is a functor from the category of spaces to itself. We will denote this functor by $X \mapsto \widehat{X}$. We think of this functor as "p-adically completing" the homotopy type of X. The following assertion makes this idea precise:

Theorem 1. *Let X be a simply connected space, and assume that every homotopy group $\pi_i X$ is finitely generated (as an abelian group). Then $\widehat{X}$ is again simply connected, and the unit map $X \to \widehat{X}$ induces isomorphisms*

$$\pi_i X \otimes_{\mathbf{Z}} \mathbf{Z}_p \simeq \pi_i \widehat{X},$$

where $\mathbf{Z}_p$ denotes the ring of p-adic integers.

We will reduce the proof of Theorem 1 to the following calculation:

Lemma 2. *For each $i \geq 0$, the canonical map*

$$\mathrm{H}_i\, K(\mathbf{Z}, 1) \to \text{“}\varprojlim \mathrm{H}_i\, K(\mathbf{Z}/p^k\mathbf{Z}, 1)\text{”}$$

is an isomorphism in the category of pro-$\mathbf{F}_p$-vector spaces.

Proof. If $i \leq 1$, then the pro-system on the right is constant (and isomorphic to the $\mathrm{H}_i\, K(\mathbf{Z}, 1)$). If $i > 1$, then the homology group on the left vanishes, and the inverse system on the right can be identified with the system

$$\ldots \to \mathbf{F}_p \xrightarrow{0} \mathbf{F}_p \xrightarrow{0} \mathbf{F}_p,$$

which is trivial as a pro-vector space. $\square$

Corollary 3. *For each $i \geq 0$ and each $n > 0$, the canonical map*

$$\phi : \mathrm{H}_i\, K(\mathbf{Z}, n) \to \text{“}\varprojlim \mathrm{H}_i\, K(\mathbf{Z}/p^k\mathbf{Z}, n)\text{”}$$

is an isomorphism in the category of pro-$\mathbf{F}_p$-vector spaces.

Proof. We work by induction on n, the case $n = 1$ having been handled above. For every abelian group A, the Eilenberg-Moore spectral sequence has E_2-term given by

$$E_2^{a,b}(A) \simeq \mathrm{Tor}_a^{\mathrm{H}_*\, K(A, n-1)}(\mathbf{F}_p, \mathbf{F}_p)_b$$

and converges to $\mathrm{H}_* K(A,n)$. It follows from the inductive hypothesis that the canonical map

$$E_2^{a,b}(\mathbf{Z}) \to \text{“}\varprojlim E_2^{a,b}(\mathbf{Z}/p^k\mathbf{Z})\text{”}$$

induces an isomorphism of pro-vector spaces for each a, b. It follows that we get an isomorphism of pro-vector spaces at the E_∞-term. The convergence of the spectral sequence them implies that ϕ is an isomorphism of pro-vector spaces. □

Corollary 4. *For each $i \geq 0$ and each $n > 0$, the canonical map*

$$\varinjlim \mathrm{H}^* K(\mathbf{Z}/p^k\mathbf{Z}, n) \to \mathrm{H}^* K(\mathbf{Z}, n)$$

is an isomorphism of $\mathbf{F}_p$-vector spaces.

Corollary 5. *Let $X = K(\mathbf{Z}, n)$, where $n \geq 1$. Then the p-profinite completion $X^\vee$ can be identified with the formal inverse limit*

$$Y = \text{“}\varprojlim K(\mathbf{Z}/p^k\mathbf{Z}, n)\text{”}.$$

Proof. We have a canonical map $X^\vee \to Y$ of p-profinite spaces. To show that it is a homotopy equivalence, it will suffice to show that it induces an isomorphism on cohomology. This follows immediately from Corollary 4. □

Corollary 6. *If $X = K(\mathbf{Z}, n)$, then the canonical map $\widehat{X} \to K(\mathbf{Z}_p, 1)$ is a homotopy equivalence.*

The following result will allow us to promote this result to more general Eilenberg-MacLane spaces:

Lemma 7. *Let X and Y be spaces such that $\mathrm{H}^*(X; \mathbf{F}_p)$ and $\mathrm{H}^*(Y; \mathbf{F}_p)$ are finite dimensional in each degree. Then the canonical ma $\widehat{X \times Y} \to \widehat{X} \times \widehat{Y}$ is a homotopy equivalence.*

Proof. Since the functor $\varprojlim : \mathfrak{S}_p^\vee \to \mathfrak{S}$ preserves homotopy limits, it will suffice to show that the canonical map $(X \times Y)^\vee \to X^\vee \times Y^\vee$ is an equivalence of p-profinite spaces. For this, it suffices to show that this map induces an isomorphism on cohomology. In general, we have isomorphisms

$$\mathrm{H}^*(X^\vee \times Y^\vee) \simeq \mathrm{H}^*(X^\vee) \otimes \mathrm{H}^*(Y^\vee) \simeq \mathrm{H}^*(X) \otimes \mathrm{H}^*(Y)$$

If the cohomology groups of X and Y are finite dimensional in each degree, then the Kunneth theorem allows us to identify this tensor product with $\mathrm{H}^*(X \times Y) \simeq \mathrm{H}^*((X \times Y)^\vee)$, as desired. □

Corollary 8. *Let A be a finitely generated abelian group and $n \geq 1$. Set $A^\vee = A \otimes_{\mathbf{Z}} \mathbf{Z}_p$. Then the canonical map $\widehat{K(A,n)} \to K(A^\vee, n)$ is a homotopy equivalence.*

Proof. Using Lemma 7 and the structure theory for finitely generated abelian groups, we can assume either that $A = \mathbf{Z}$ or that $A \simeq \mathbf{Z}/l^k\mathbf{Z}$, where l is some prime number. In the first case, the desired result follows from Corollary 6. If $l = p$, then $K(A, n) = K(A^\vee, n)$ is p-finite and the result is obvious. If l is distinct from p, then $K(A, n)$ has trivial cohomology (with coefficients in $\mathbf{F}_p$), so that $\widehat{K(A,n)}$ and $K(A^\vee, n)$ are both contractible. □

Lemma 9. *Suppose given a homotopy pullback square*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

of simply connected spaces, whose cohomology groups (with coefficients in $\mathbf{F}_p$) are finite dimensional in each degree. Then the induced square

$$\begin{array}{ccc} \widehat{X'} & \longrightarrow & \widehat{X} \\ \downarrow & & \downarrow \\ \widehat{Y'} & \longrightarrow & \widehat{Y} \end{array}$$

is a homotopy pullback diagram.

Proof. As before, it suffices to show that the diagram

$$\begin{array}{ccc} X'^{\vee} & \longrightarrow & X^{\vee} \\ \downarrow & & \downarrow \\ Y'^{\vee} & \longrightarrow & Y^{\vee} \end{array}$$

is a homotopy pullback diagram of p-profinite spaces, which is equivalent to the assertion that the diagram

$$\begin{array}{ccc} C^*(X') & \longleftarrow & C^*(X) \\ \uparrow & & \uparrow \\ C^*(Y') & \longleftarrow & C^*(Y) \end{array}$$

is a homotopy pushout diagram of E_∞-algebras over $\mathbf{F}_p$. This is equivalent to the convergence of the cohomological Eilenberg-Moore spectral sequence; we proved this result in the case where all of the spaces involved were p-finite. However, our proof only used the finite dimensionality of cohomology groups and the nilpotence of the spaces involved; in particular, it remains valid when each space is simply connected and has cohomology of finite type. □

We are now ready to prove our main result:

Proof of Theorem 1. Let X be a simply connected space whose homotopy groups are finitely generated. Then X has a Postnikov tower

$$\ldots \to \tau_{\leq 3} X \to \tau_{\leq 2} X \to \tau_{\leq 1} X \simeq *,$$

where $\tau_{\leq n} X$ is obtained from X by killing the homotopy groups of X above dimension n. In particular, the map $X \to \tau_{\leq n} X$ is highly connected if n is large, so that $\mathrm{H}^* X \simeq \varinjlim \mathrm{H}^* \tau_{\leq n} X$. It follows that we have an equivalence of p-profinite spaces

$$X^{\vee} \simeq \varprojlim (\tau_{\leq n} X)^{\vee}.$$

Passing to the homotopy inverse limit, we get a homotopy equivalence

$$\widehat{X} \simeq \varprojlim \widehat{\tau_{\leq n} X}.$$

It will therefore suffice to prove the analogous result after replacing X by $\tau_{\leq n} X$. We now proceed by induction on n, using the existence of a homotopy pullback square

$$\begin{array}{ccc} \tau_{\leq n} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ \tau_{\leq n-1} X & \longrightarrow & K(\pi_n X, n+1). \end{array}$$

The desired result now follows by combining the inductive hypothesis, Lemma 9, and Corollary 8. □

We conclude this section by giving a characterization of $\widehat{X}$ by a universal property. We first recall Bousfield's notion of an $\mathbf{F}_p$-*local space*.

Definition 10. A map $f : X \to Y$ of spaces is said to be an $\mathbf{F}_p$-*equivalence* if the induced map on cohomology $\mathrm{H}^*(Y) \to \mathrm{H}^*(X)$ is an isomorphism.

A space Z is said to be $\mathbf{F}_p$-*local* if, for every $\mathbf{F}_p$-equivalence $f : X \to Y$, the induced map $\mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$ is a homotopy equivalence.

Example 11. Every Eilenberg-MacLane space $K(\mathbf{F}_p, n)$ is $\mathbf{F}_p$-local (since the homotopy groups of the mapping space $\mathrm{Map}(X, K(\mathbf{F}_p, n))$ can be identified with cohomology groups of X with coefficients in $\mathbf{F}_p$).

It is clear that the collection of $\mathbf{F}_p$-local spaces is closed under homotopy limits. Since every p-finite space X can be built from Eilenberg-MacLane spaces $K(\mathbf{F}_p, n)$ using finite homotopy limits, we conclude that p-finite spaces are $\mathbf{F}_p$-local. It follows that any homotopy limit of p-finite spaces is again $\mathbf{F}_p$-local. In particular, for *any* space X, the space $\widehat{X} = \varprojlim X^\vee$ is $\mathbf{F}_p$-local.

Definition 12. We say that a map of spaces $f : X \to X'$ *exhibits* X' *as an* $\mathbf{F}_p$-*localization of* X if f is an $\mathbf{F}_p$-equivalence and X' is $\mathbf{F}_p$-local.

Remark 13. For any space X, there exists an $\mathbf{F}_p$-localization X' of X, and X' is uniquely determined up to weak homotopy equivalence.

Proposition 14. *Let X be a simply connected space whose homotopy groups are finitely generated. Then the unit map $f : X \to \widehat{X}$ exhibits $\widehat{X}$ as an $\mathbf{F}_p$-localization of X.*

Proof. We have seen above that $\widehat{X}$ is $\mathbf{F}_p$-local. It will therefore suffice to show that f induces an isomorphism on cohomology with coefficients modulo p. Using the Serre spectral sequence repeatedly, we can reduce to the case where X is an Eilenberg-MacLane space $K(A, n)$, where A is a finitely generated abelian group. Then $\widehat{X} = K(A^\vee, n)$. We then have a fiber sequence

$$X \to \widehat{X} \to K(A^\vee/A, n).$$

Using the Serre spectral sequence again, it will suffice to show that the space $K(A^\vee/A, n)$ has trivial cohomology with coefficients in $\mathbf{F}_p$. We can then invoke the following Lemma:

Lemma 15. *Let B be an abelian group such that multiplication by p is an isomorphism from B to itself, and let $n \geq 1$. Then $\mathrm{H}_* K(B, n)$ vanishes for $* > 0$.*

Proof. Since the functor $B \mapsto \mathrm{H}_* K(B, n)$ commutes with filtered colimits, we may assume without loss of generality that B is a finitely generated module over $\mathbf{Z}[\frac{1}{p}]$. Using the Eilenberg-Moore spectral sequence, we can assume $n = 1$. Using the structure theorem for finitely generated abelian groups and the Kunneth formula, we may assume either that $B = \mathbf{Z}[\frac{1}{p}]$ or that $B = \mathbf{Z}/l^k\mathbf{Z}$, where $l \neq p$. In the second case the result is clear: the homology of a finite group G is always trivial at any prime which does not divide the order $|G|$. In the first case, $K(B, 1)$ is the homotopy colimit of the sequence

$$S^1 \xrightarrow{p} S^1 \xrightarrow{p} S^1 \to \dots,$$

so we have $\mathrm{H}_* K(B, 1) \simeq \varinjlim \mathrm{H}_* S^1$ and the result follows by inspection. □

□

Remark 16. For a general space X, the unit map $X \to \widehat{X}$ need *not* induce an isomorphism on $\mathbf{F}_p$-cohomology, so that $\widehat{X}$ need not be an $\mathbf{F}_p$-localization of X.