18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

Free E_∞-Algebras (Lecture 21)

In this lecture we will review the theory of E_∞-algebras over the field $\mathbf{F}_2$ of two elements.

Roughly speaking, an E_∞-algebra over $\mathbf{F}_2$ is a chain complex V of $\mathbf{F}_2$-vector spaces, equipped with a multiplication

$$m : V \otimes V \to V$$

which is commutative, associative, and unital, up to coherent homotopy. We summarize some of the basic properties of this notion:

(1) For every topological space X, the cochain complex $C^*(X)$ has the structure of an E_∞-algebra over $\mathbf{F}_2$.

(2) If V is an E_∞-algebra over $\mathbf{F}_2$, then the product map m descends to a good symmetric multiplication $D_2(V) \to V$, in the sense of our previous lectures. Consequently, the cohomology $\mathrm{H}^*(V)$ is endowed with the structure of an unstable $\mathcal{A}^{\mathrm{Big}}$-module, where $\mathcal{A}^{\mathrm{Big}}$ denotes the big Steenrod algebra.

(3) The forgetful functor

$$\{E_\infty - \text{algebras over } \mathbf{F}_2\} \to \{\text{chain complexes over } \mathbf{F}_2\}$$

admits a left adjoint $\mathcal{F}$. The functor $\mathcal{F}$ carries a chain complex V to the symmetric algebra

$$\mathcal{F}(V) \oplus_{n \geq 0} V^{\otimes n}_{h\Sigma_n} = \oplus_{n \geq 0} D_n(V),$$

where D_n denotes the nth extended power functor.

For every integer n, we let $\mathcal{F}(n) = \mathcal{F}(\mathbf{F}_2[-n])$ denote the free E_∞-algebra over $\mathbf{F}_2$ generated by a single class of cohomological degree n. By construction, we have a canonical map of complexes

$$\mathbf{F}_2[-n] \to \mathcal{F}(n),$$

which determines an element $\eta \in \mathrm{H}^n\, \mathcal{F}(n)$. Since $\mathrm{H}^*\, \mathcal{F}(n)$ has the structure of an unstable $\mathcal{A}^{\mathrm{Big}}$-algebra, the element η determines a map

$$\theta_n : F^{\mathrm{Big}}_{\mathrm{Alg}}(n) \to \mathrm{H}^*\, \mathcal{F}(n).$$

Here $F^{\mathrm{Big}}_{\mathrm{Alg}}(n)$ denotes the free unstable $\mathcal{A}^{\mathrm{Big}}$-module on one generator μ_n in degree n, whose structure was determined in Lecture 11.

Our goal in this lecture is to prove the following result:

Theorem 1. *For every integer n, the map θ_n is an isomorphism.*

To prove Theorem 1, we will show by two separate arguments that θ_n is injective and that θ_n is surjective. We begin with the injectivity. Recall that $F^{\mathrm{Big}}_{\mathrm{Alg}}(n)$ has a basis consisting of expressions

$$\{\mathrm{Sq}^{I_1}(\mu_n)\, \mathrm{Sq}^{I_2}(\mu_n) \ldots \mathrm{Sq}^{I_k}(\mu_n)\},$$

where $I_1, \ldots, I_k$ range over distinct admissible sequences of excess $\leq n$. This module has a grading by cohomological degree, but also another grading by rank, where we declare

$$\mathrm{rk}(1) = 0$$

$$\mathrm{rk}(\mu_n) = 1$$

$$\mathrm{rk}(xy) = \mathrm{rk}(x) + \mathrm{rk}(y)$$

$$\mathrm{rk}(\mathrm{Sq}^i(x)) = 2\,\mathrm{rk}(x).$$

Similarly, the cohomology $\mathrm{H}^*\,\mathcal{F}(n)$ can be written as a direct sum

$$\oplus_{k\geq 0}\,\mathrm{H}^*\,D_k(\mathbf{F}_2[-n])$$

is equipped with a grading by rank, where elements $\mathrm{H}^*\,D_k(\mathbf{F}_2[-n])$ have rank k. The multiplication on $\mathcal{F}(n)$ carries $D_k(\mathbf{F}_2[-n]) \otimes D_{k'}(\mathbf{F}_2[-n])$ into $D_{k+k'}(\mathbf{F}_2[-n])$, and Steenrod operations Sq^i carry $\mathrm{H}^*\,D_k(\mathbf{F}_2[-n])$ into $\mathrm{H}^{*+i}\,D_{2k}(\mathbf{F}_2[-n])$. It follows that the map θ_n is compatible with the grading by rank.

Recall that we defined shift isomorphisms

$$S : F^{\mathrm{Big}}_{\mathrm{Alg}}(n) \to F^{\mathrm{Big}}_{\mathrm{Alg}}(n+1).$$

The map S is an isomorphism of commutative rings (not compatible with the action of $\mathcal{A}^{\mathrm{Big}}$), which is uniquely determined by the following requirements:

$$S(\mu_n) = \mu_{n+1}$$

$$S(\mathrm{Sq}^i(x)) = \mathrm{Sq}^{i+\mathrm{rk}(x)}\,S(x).$$

The shift maps S do not respect degree, but instead satisfy the formula

$$\deg(Sx) = \deg(x) + \mathrm{rk}(x)$$

whenever x is homogeneous in both degree and rank.

We have similar isomorphisms $S' : \mathrm{H}^*\,\mathcal{F}(n) \to \mathrm{H}^*\,\mathcal{F}(n+1)$, obtained by taking the direct sum of the canonical isomorphisms

$$\mathrm{H}^*\,D_k(\mathbf{F}_2[-n]) = \mathrm{H}^{*-nk}(B\Sigma_k, \mathbf{F}_2) \simeq \mathrm{H}^{*+k}\,D_k(\mathbf{F}_2[-n-1]).$$

For every integer n, we have a commutative diagram

$$\begin{array}{ccc} F^{\mathrm{Big}}_{\mathrm{Alg}}(n) & \xrightarrow{S} & F^{\mathrm{Big}}_{\mathrm{Alg}}(n+1) \\ \downarrow{\scriptstyle \theta_n} & & \downarrow{\scriptstyle \theta_{n+1}} \\ \mathrm{H}^*\,\mathcal{F}(n) & \xrightarrow{S'} & \mathrm{H}^*\,\mathcal{F}(n+1), \end{array}$$

We are now ready to prove injectivity of θ_n. Suppose that θ_n fails to be injective. Choose some nonzero element

$$x = \sum_{\alpha} \mathrm{Sq}^{I^{\alpha}_1}(\mu_n)\ldots\mathrm{Sq}^{I^{\alpha}_{k_\alpha}}(\mu_n)$$

in the kernel of θ^n, where the sequences I^{α}_i are admissible, distinct (for fixed α), and have excess $\leq n$. Then for every integer $p \geq 0$, the element

$$S^p(x) = \sum_{\alpha} \mathrm{Sq}^{J^{\alpha}_1}(\mu_{n+p})\ldots\mathrm{Sq}^{J^{\alpha}_{k_\alpha}}(\mu_{n+p})$$

lies in the kernel of θ_{n+p}. Choosing $p \gg 0$ and replacing x by $S^p(x)$, we may assume that each of the sequences I_i^α is positive. It follows that the image of x in the free algebra $F_{\mathrm{Alg}}(n)$ is nonzero. But the Cartan-Serre theorem identifies $F_{\mathrm{Alg}}(n)$ with the cohomology ring

$$\mathrm{H}^* K(\mathbf{F}_2, n),$$

which is the cohomology of the E_∞-algebra $C^*K(\mathbf{F}_2, n)$. The universal property of $\mathcal{F}(n)$ gives a map of E_∞-algebra $\mathcal{F}(n) \to C^*K(\mathbf{F}_2, n)$, which fits into a commutative diagram

$$\begin{array}{ccc} F_{\mathrm{Alg}}^{\mathrm{Big}}(n) & \longrightarrow & \mathrm{H}^* K(\mathbf{F}_2, n) \\ & {\scriptstyle \theta_n} \searrow \quad \nearrow & \\ & \mathrm{H}^* \mathcal{F}(n). & \end{array}$$

It follows that $\theta_n(x) \neq 0$, a contradiction.

We now prove the surjectivity of θ_n. The proof is based on the following elementary lemma:

Lemma 2. *Let $H \subseteq G$ be finite groups, and suppose that $|G/H|$ is odd. Then the induced map on homology*

$$p : \mathrm{H}_*(BH) \to \mathrm{H}_*(BG)$$

is an isomorphism.

Proof. We can realize the map of classifying spaces $BH \to BG$ as a covering space map, whose fiber has cardinality $|G/H|$. We therefore have a transfer map

$$t : \mathrm{H}_*(BG) \to \mathrm{H}_*(BH).$$

The composition $p \circ t$ is given by multiplication by $|G/H|$, and is therefore an isomorphism. Since $p \circ t$ is surjective, the map p must also be surjective. □

We now return to the proof of Theorem 1. We will show, by induction on $k \geq 0$, that the map

$$\theta_n : F_{\mathrm{Alg}}^{\mathrm{Big}}(n)_k \to \mathrm{H}^* \mathcal{F}(n)_k = \mathrm{H}^* D_k(\mathbf{F}_2[-n])$$

is surjective; here the subscripts indicate that we consider only the component consisting of elements of rank k. If $k = 0$, this is clear: the only element of rank 0 on the right hand side is the unit 1, and we have $\theta_n(1) = 1$. Similarly, the only element of rank 1 on the right hand side is the generator $\eta \in \mathrm{H}^n \mathcal{F}(n)$, and we have $\theta_n(\mu_n) = \eta$ by construction. We may therefore assume that $k > 1$. There are two cases to consider:

- Suppose that k is not a power of 2. Then we can write $k = k' + k''$, where $\binom{k}{k'} = \frac{k!}{k'!k''!}$ is odd. Multiplication yields a commutative diagram

$$\begin{array}{ccc} F_{\mathrm{Alg}}^{\mathrm{Big}}(n)_{k'} \otimes F_{\mathrm{Alg}}^{\mathrm{Big}}(n)_{k''} & \longrightarrow & F_{\mathrm{Alg}}^{\mathrm{Big}}(n)_k \\ \downarrow {\scriptstyle \theta_n \otimes \theta_n} & & \downarrow {\scriptstyle \theta_n} \\ \mathrm{H}^* D_{k'}(\mathbf{F}_2[-n]) \otimes \mathrm{H}^* D_{k''}(\mathbf{F}_2[-n]) & \longrightarrow & \mathrm{H}^* D_k(\mathbf{F}_2[-n]). \end{array}$$

  The inductive hypothesis guarantees that the left vertical map is surjective. To prove that the right vertical map is surjective, it will suffice to show that the lower horizontal map is surjective. Up to a shift, this agrees with the pushforward map

$$\mathrm{H}_*(B(\Sigma_{k'} \times \Sigma_{k''})) \to \mathrm{H}_*(B\Sigma_k),$$

which is surjective by Lemma 2 since

$$|\Sigma_k/(\Sigma_{k'} \times \Sigma_{k''})| = \frac{k!}{k'!k''!}$$

is odd by assumption.

- Suppose that k is a power of 2, and let $k' = \frac{k}{2}$. We have a map of extended powers

  $$D_2 D_{k'} \mathbf{F}_2[-n] \to D_k \mathbf{F}_2[-n].$$

  Up to a shift, the induced map on cohomology can be identified with the map

  $$p : \mathrm{H}_*(BG) \to \mathrm{H}_*(B\Sigma_k),$$

  where $G \subset \Sigma_k$ is the wreath product $\Sigma_{k'}^2 \rtimes \Sigma_2$. We observe that $|\Sigma_k/G|$ is odd, so the map p is surjective by Lemma 2.

Recall that if V is a complex of $\mathbf{F}_2$-vector spaces such that the cohomology $\mathrm{H}^* V$ has a basis $\{v_i\}$, then the cohomology $\mathrm{H}^* D_2(V)$ has a basis consisting of pairwise products $\{v_i v_j\}_{i<j}$, together with Steenrod operations $\{\overline{\mathrm{Sq}}\, v_i\}$. It follows that $\mathrm{H}^* D_k \mathbf{F}_2[-n]$ is generated by $\mathrm{H}^* D_{k'} \mathbf{F}_2[-n]$ under the operations of pairwise product and Steenrod operations Sq^i. The map θ_n is a map of unstable $\mathcal{A}^{\mathrm{Big}}$-algebras, so the image of θ_n is stable under the formation of products and closed under the operations Sq^i. The inductive hypothesis implies that $\mathrm{H}^* D_{k'} \mathbf{F}_2[-n]$ belongs to the image of θ_n, so that $\mathrm{H}^* D_k \mathbf{F}_2[-n]$ belongs to the image of θ_n as well.