18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

Free Unstable Algebras (Lecture 12)

In the last lecture, we introduced the free unstable $\mathcal{A}$-module $\mathrm{F}_{\mathrm{Alg}}(n)$ on a generator μ_n having degree n. Moreover, we asserted the following:

Proposition 1. *For $n \geq 0$, the vector space $\mathrm{F}_{\mathrm{Alg}}(n)$ has a basis consisting of*

$$\{\mathrm{Sq}^{I_1}(\mu_n) \ldots \mathrm{Sq}^{I_k}(\mu_n)\},$$

where $I_1 < \ldots < I_k$ are admissible positive sequences of excess $\leq n$. Here we adopt the convention that the excess of the empty sequence is $-\infty$.

Moreover, we have already proven half of this result: namely, that the products $\mathrm{Sq}^{I_1}(\mu_n) \ldots \mathrm{Sq}^{I_k}(\mu_n)$ generate $\mathrm{F}_{\mathrm{Alg}}(n)$. To complete the proof, we must show that these elements are linearly independent. For this, it will suffice to construct a map $\phi : \mathrm{F}_{\mathrm{Alg}}(n) \to M$, where M is an unstable $\mathcal{A}$-algebra, such that the images $\phi(\mathrm{Sq}^{I_1}(\mu_n) \ldots \mathrm{Sq}^{I_k}(\mu_n))$ are linearly independent in M. Since $\mathrm{F}_{\mathrm{Alg}}(n)$ is freely generated by μ_n, the map ϕ is determined by a single element $x = \phi(\mu_n) \in M^n$.

We have seen that for every topological space X, the cohomology ring $\mathrm{H}^*(X)$ is an unstable $\mathcal{A}$-algebra. It is therefore natural to try to prove Proposition 1 by producing a space X and a cohomology class $x \in \mathrm{H}^n(X)$ such that the elements $\{\mathrm{Sq}^{I_1}(x) \ldots \mathrm{Sq}^{I_k}(x)\}$ are linearly independent in $\mathrm{H}^*(X)$. To guarantee this, we want to choose X such that the cohomology class is as nontrivial as possible. There is a natural candidate for X: namely, the Eilenberg-MacLane space $K(\mathbf{F}_2, n)$. This Eilenberg-MacLane space *represents* the cohomology functor $X \mapsto \mathrm{H}^n(X)$ in the following sense: there is a canonical element $\chi \in \mathrm{H}^n(K(\mathbf{F}_2, n))$, and for every nice topological space, the pullback of χ induces a bijection

$$[X, K(\mathbf{F}_2, n)] \simeq \mathrm{H}^n(X).$$

Here $[X, K(\mathbf{F}_2, n)]$ denotes the set of homotopy classes of maps from X to $K(\mathbf{F}_2, n)$. Consequently, if we hope to prove Proposition 1 using the unstable $\mathcal{A}$-algebras provided by the cohomology of *any* space X, then we might as well replace X by $K(\mathbf{F}_2, n)$. Fortunately, this turns out to work. More precisely, Proposition **??** is a consequence of the following result:

Theorem 2 (Cartan, Serre)**.** *For each $n \geq 0$, the cohomology ring $\mathrm{H}^*(K(\mathbf{F}_2, n))$ has a basis $\{\mathrm{Sq}^{I_1}(\chi) \ldots \mathrm{Sq}^{I_k}(\chi)\}$, where $I_1 < \ldots < I_k$ are admissible positive sequences of excess $\leq n$.*

Corollary 3. *The canonical map $\phi : \mathrm{F}_{\mathrm{Alg}}(n) \to \mathrm{H}^*(K(\mathbf{F}_2, n))$ is an isomorphism.*

To put Corollary 3 in perspective, let us recall a definition. A *cohomology operation* is a collection of maps

$$\mathrm{H}^n(X) \to \mathrm{H}^m(X),$$

defined for all topological spaces X and functorial in X. For example, every Steenrod operation Sq^i determines a cohomology operation

$$\mathrm{Sq}^i : \mathrm{H}^n(X) \to \mathrm{H}^{n+i}(X).$$

Using Yoneda's lemma, we see that the set of cohomology operations from H^n to H^m can be identified with

$$[K(\mathbf{F}_2, n), K(\mathbf{F}_2, m)] \simeq \mathrm{H}^m(K(\mathbf{F}_2(n))) \simeq \mathrm{F}_{\mathrm{Alg}}(n)^m.$$

In other words, we can build *every* cohomology operation out of Steenrod squares, sums, and products. Moreover, the only relations among these operations are the ones we have built into the definition of an unstable $\mathcal{A}$-algebra:

(i) The Adem relations

$$\mathrm{Sq}^a \mathrm{Sq}^b(x) = \sum_k (2k-a, b-k-1) \mathrm{Sq}^{b+k} \mathrm{Sq}^{a-k}(x)$$

for $a < 2b$.

(ii) The Cartan formula $\mathrm{Sq}^n(xy) = \sum_{n=n'+n''} \mathrm{Sq}^{n'}(x) \mathrm{Sq}^{n''}(y)$.

(iii) The boundary conditions

$$\mathrm{Sq}^n(x) = \begin{cases} 0 & \text{if } n < 0 \\ x & \text{if } n = 0 \\ ? & \text{if } 0 < n < \deg(x) \\ x^2 & \text{if } n = \deg(x) \\ 0 & \text{if } n > \deg(x). \end{cases}$$

We will later show that there is an analogous relationship between unstable $\mathcal{A}^{\mathrm{Big}}$-modules and the cohomology of E_∞-algebras over $\mathbf{F}_2$.

We now turn to the proof of Theorem 2. We begin by modifying the formulation a bit. Recall that the excess of a sequence of integers $I = (i_m, \ldots, i_0)$ is the difference

$$i_m - i_{m-1} - \ldots - i_0 = (i_m - 2i_{m-1}) + \ldots + (i_1 - 2i_0) + i_0.$$

This definition is rigged so that if I has excess $> \deg(x)$, then $\mathrm{Sq}^I(x) = \mathrm{Sq}^{i_m}(\mathrm{Sq}^{I'} x) = 0$, where $I' = (i_{m-1}, \ldots, i_0)$, since

$$i_m > i_{m-1} + \ldots + i_0 + \deg(x) = \deg(\mathrm{Sq}^{I'} x).$$

If the excess of I is *exactly* $\deg(x)$, then we instead have the equality $\mathrm{Sq}^I(x) = \mathrm{Sq}^{i_m} \mathrm{Sq}^{I'}(x) = (\mathrm{Sq}^{I'} x)^2$.

Applying this argument repeatedly, we see that every expression $\mathrm{Sq}^{I_1}(\chi) \ldots \mathrm{Sq}^{I_k}(\chi)$ appearing in Theorem 2 can be rewritten uniquely as a product $(\mathrm{Sq}^{I'_1}(\chi))^{2^{a_1}} \ldots (\mathrm{Sq}^{I'_k}(\chi))^{2^{a_k}}$, where each I'_j is an admissible positive sequence of excess $< n$, and the a_j are nonnegative integers, and the pairs (a_j, I'_j) are disjoint. Since every nonnegative integer b has a unique expansion as a sum of distinct powers of 2, we obtain the following reformulation of the Cartan-Serre theorem:

Theorem 4. *For $n \geq 0$. The cohomology ring $\mathrm{H}^*(K(\mathbf{F}_2, n))$ has a basis consisting of products $\{\mathrm{Sq}^{J_1}(\chi)^{b_1} \ldots \mathrm{Sq}^{J_k}(\chi)^{b_k}\}$, where $J_1 < \ldots < J_k$ are admissible positive sequences of excess $< n$, and the b_j are nonnegative integers. In other words, $\mathrm{H}^*(K(\mathbf{F}_2, n))$ is a polynomial ring on generators $\{\mathrm{Sq}^J(\chi)\}$, where J ranges over admissible positive sequences of excess $< n$.*

We now turn to the proof of this theorem. The case $n = 0$ is trivial. To handle the case $n = 1$, we observe that every nonempty positive admissible sequence $(i_n, \ldots, i_0)$ has positive excess. Thus, there is only one sequence with excess < 1: the empty sequence J (which, by convention, has excess $-\infty$). We have $\mathrm{Sq}^J(\chi) = \chi$, and Theorem 4 reduces to the following assertion: the cohomology ring $\mathrm{H}^*(K(\mathbf{F}_2, 1))$ is a polynomial ring on its canonical element $\chi \in \mathrm{H}^1(K(\mathbf{F}_2, 1))$. But $K(\mathbf{F}_2, 1)$ is simply the classifying space $B\Sigma_2 \simeq \mathbf{R}P^\infty$, whose cohomology ring is indeed isomorphic to a polynomial ring $\mathbf{F}_2[t]$ on a single generator.

To treat the general case, we will use induction on n and the Serre spectral sequence. We begin by reviewing the Serre spectral sequence in general.

Fact 5 (Serre)**.** *Suppose given a homotopy fiber sequence of topological spaces*

$$F \to E \to B.$$

Then there exists a (first quadrant) spectral sequence

$$\{E_r^{p,q}, d_r\}_{r \geq 2}$$

with $E_2^{p,q} \simeq \mathrm{H}^p(B; \mathrm{H}^q(F; \mathbf{F}_2))$ *which converges to the cohomology* $\mathrm{H}^{p+q}(E; \mathbf{F}_2)$*. Moreover:*

(1) $\{E_r^{p,q}, d_r\}_{r\geq 2}$ *is a spectral sequence of* algebras.

(2) *If the base* B *is simply connected and the cohomology groups* $\mathrm{H}^q(F; \mathbf{F}_2)$ *are finite dimensional, then obtain a canonical isomorphism* $E_2^{p,q} \simeq \mathrm{H}^p(B) \otimes \mathrm{H}^q(F)$*.*

Since the Serre spectral sequence $\{E_r^{p,q}, d_r\}_{r\geq 2}$ is a first quadrant spectral sequence, we see that for each $r \geq 2$, the groups $E_r^{0,q}$ can be identified with subgroups of $E_2^{0,q}$ (namely, the subgroups consisting of elements killed by the differentials $d_2, \ldots, d_{r-1}$), and the groups $E_r^{p,0}$ can be identified with quotients of $E_r^{p,0}$ (the quotient by the images of the differentials $d_2, \ldots, d_{r-1}$).

We will be interested in studying the Serre spectral sequence in the case where the total space E of the fibration is contractible. In this case, we deduce that $E_\infty^{p,q} \simeq 0$ unless $p = q = 0$. In particular, for each $m \geq 2$ the "final differential" $d_m : E_m^{0,m-1} \to E_m^{m,0}$ must be an isomorphism. The composition

$$\tau : \mathrm{H}^m(B) \simeq E_2^{m,0} \to E_m^{m,0} \stackrel{d_m^{-1}}{\to} E_m^{0,m-1} \subseteq E_2^{0,m-1} \simeq \mathrm{H}^{m-1}(F)$$

is called the *trangression map*. Elements of $\mathrm{H}^{m-1}(F)$ which lie in the image of τ are called *transgressive*.

There is a canonical example of a spectral sequence with a trangressive element x of degree $m-1$, which we will denote by $\{E(m)_r^{p,q}, d_r\}_{r\geq 2}$. Namely, we take

$$E(m)_r^{p,q} = \begin{cases} \mathbf{F}_2[y] \oplus \mathbf{F}_2[y]x & \text{if } r \leq m \\ \mathbf{F}_2 & \text{if } r > m \end{cases}$$

where x has degree $(0, m-1)$ and y has degree $(m, 0)$. The differentials d_r vanish unless $r = m$, and d_r is given by the formula

$$d_m(z) = \begin{cases} 0 & \text{if } z = y^a \\ y^{a+1} & \text{if } z = y^a x. \end{cases}$$

In this spectral sequence, the transgression map carries y to x, and vanishes in other degrees. Moreover, given any spectral sequence of algebras $\{E_r^{p,q}, d_r\}_{r\geq 2}$ with a transgressive element $\chi' = \tau(\chi) \in E_2^{0,m-1}$, there is a unique map of spectral sequences $\{E(m)_r^{p,q}\} \to \{E_r^{p,q}\}$, which is given by the formula $y^a \mapsto \chi^a$, $y^a x \mapsto \chi^a \chi'$.

Let us now return to our discussion of the Serre spectral sequence of Fact 5, where we can describe the transgression map in topological terms. If we assume that the total space E of the fibration is contractible, then we can identify the fiber F with the based loop space ΩB. We then have a canonical map $\Sigma\Omega B \to B$, which induces a pullback map on reduced cohomology

$$\tau : \widetilde{\mathrm{H}}^*(B) \to \widetilde{\mathrm{H}}^*(\Sigma\Omega B) \simeq \widetilde{\mathrm{H}}^{*-1}(\Omega B) = \widetilde{\mathrm{H}}^{*-1}(F).$$

From this description of the transgression map (and the stability of the Steenrod operations), we see that τ commutes with the action of the Steenrod operations.

We now specialize to the case of interest: let the base B be the Eilenberg-MacLane space $K(\mathbf{F}_2, n)$, where $n \geq 2$. We will take the map $E \to B$ to be the usual path fibration, so that E is contractible and the fiber F is isomorphic to $\Omega K(\mathbf{F}_2, n) \simeq K(\mathbf{F}_2, n-1)$. Let $\chi \in \mathrm{H}^n(K(\mathbf{F}_2, n))$ be the canonical generator, and let $\chi' = \tau(\chi) \in \mathrm{H}^{n-1}(K(\mathbf{F}_2, n-1))$. For every positive admissible sequence I of excess $< n$, the element $\mathrm{Sq}^I \chi' \in \mathrm{H}^*(K(\mathbf{F}_2, n-1))$ is the image of $\mathrm{Sq}^I \chi$ under the transgression map. It follows from the above that we get a map of spectral sequences

$$\psi_I : \{E(n + \deg(I))_r^{p,q}\}_{r \geq 2} \to \{E_r^{p,q}\}_{r \geq 2},$$

given by $y \mapsto \chi$, $x \mapsto \chi'$.

Let $\{\widetilde{E}_r^{p,q}\}_{r \geq 2}$ denote the tensor product of the spectral sequences $\{E(n + \deg(I))_r^{p,q}\}$, taken over all positive admissible sequences I of excess $< n$. Since the Serre spectral sequence $\{E_r^{p,q}\}_{r \geq 2}$ is a spectral sequence of commutative algebras, we can multiply the maps ψ_I to obtain a single map

$$\psi : \{\widetilde{E}_r^{p,q}\}_{r \geq 2} \to \{E_r^{p,q}\}_{r \geq 2}.$$

We now make the following observations:

(a) The map ψ induces an isomorphism of columns

$$\widetilde{E}_2^{0,*} \to E_2^{0,*} \simeq \mathrm{H}^*(K(\mathbf{F}_2, n-1)).$$

This is simply a reformulation of Theorem 2 for the Eilenberg-MacLane space $K(\mathbf{F}_2, n-1)$, which follows from our inductive hypothesis.

(b) The spectral sequence $\{\widetilde{E}_r^{p,q}\}_{r \geq 2}$ is a spectral sequence of modules over the ring $R = \widetilde{E}_2^{*,0}$, which is a polynomial ring on a set of generators $y(I)$, where I ranges over admissible positive sequences of excess $\leq n$.

(c) The spectral sequence $\{E_r^{p,q}\}_{r \geq 2}$ is a spectral sequence of modules over the ring $\mathrm{H}^*(B) \simeq E_2^{*,0}$. Moreover, the map ψ induces a ring homomorphism $R \to \mathrm{H}^*(B)$ which carries $y(I)$ to $\mathrm{Sq}^I \chi$.

(d) For each $q \geq 0$, $\widetilde{E}_2^{*,q}$ is freely generated by $\widetilde{E}_2^{0,q}$ as an R-module. Similarly, $E_2^{*,q}$ is freely generated by the same vector space $E_2^{0,q} \simeq \widetilde{E}_2^{0,q}$ as an $\mathrm{H}^*(B)$-module.

(e) The map ψ induces an isomorphism $\widetilde{E}_\infty^{p,q} \to E_\infty^{p,q}$, since both sides vanish unless $p = q = 0$.

To prove Theorem 4, we must show that the map $R \to \mathrm{H}^*(B)$ is an isomorphism of rings. In fact, we will prove the stronger assertion that ψ is an isomorphism of spectral sequences. It will suffice to show that ψ induces an isomorphism $\widetilde{E}_2^{p,q} \to E_2^{p,q}$ for $p, q \geq 0$. The proof is by induction on p. If $p = 0$, the desired result follows from (a).

Suppose $p > 0$. In view of (d), it will suffice to show that ψ induces an isomorphism $\widetilde{E}_2^{p,0} \to E_2^{p,0}$. For $q \leq p - 1$, let $D(q)$ denote the quotient of $E_{q+1}^{p-q-1,q}$ by the images of the maps $\{d_r\}_{r \geq q+1}$, and let $\widetilde{D}(q)$ be defined likewise. Since $E_\infty^{p,0} \simeq 0$, we conclude that $E_2^{p,0}$ admits a finite filtration whose successive quotients are the vector spaces $\{D(q)\}_{0 \leq q \leq p-1}$. Similarly, $\widetilde{E}_2^{p,0}$ admits a filtration with successive quotients $\{\widetilde{D}(q)\}_{0 \leq q \leq p-1}$. Using the inductive hypothesis, we see that ψ induces an isomorphism $\widetilde{D}(q) \to D(q)$. It follows that ψ also induces an isomorphism $\widetilde{E}_2^{p,0} \to E_2^{p,0}$ as desired.