37 Poincaré duality

Let M be a n-manifold and K a compact subset. By Theorem 32.1

$$H_n(M, M-K; R) \xrightarrow{\cong} \Gamma(K; o_M \otimes R)\,.$$

An *orientation along* K is a section of $o_M \otimes R$ over K that restricts to a generator of $H_n(M, M-x; R)$ for every $x \in K$. The corresponding class in $H_n(M, M-K; R)$ is a *fundamental class along* K, $[M]_K$. We recall also the fully relative cap product pairing (in which $p+q=n$ and L is a closed subset of K)

$$\cap : \check{H}^p(K, L; R) \otimes_R H_n(M, M-K; R) \to H_q(M-L, M-K; R)\,.$$

We now combine all of this in the following climactic result.

Theorem 37.1 (Fully relative Poincaré duality)**.** *Let M be an n-manifold and $K \supseteq L$ a pair of compact subsets. Assume given an R-orientation along K, with corresponding fundamental class $[M]_K$. With $p+q=n$, the map*

$$\cap[M]_K : \check{H}^p(K, L; R) \to H_q(M-L, M-K; R)\,.$$

is an isomorphism.

We have seen that these isomorphisms are compatible; they form the rungs of the commuting ladder

$$\begin{array}{ccccccccccc}
\cdots \to & \check{H}^{p-1}(L) & \to & \check{H}^p(K,L) & \to & \check{H}^p(K) & \to & \check{H}^p(L) & \to \cdots \\
& \downarrow{\scriptstyle \cap[M]_L} & & \downarrow{\scriptstyle \cap[M]_K} & & \downarrow{\scriptstyle \cap[M]_K} & & \downarrow{\scriptstyle \cap[M]_L} & \\
\cdots \to & H_{q+1}(M, M-L) & \to & H_q(M-L, M-K) & \to & H_q(M, M-K) & \to & H_q(M, M-L) & \to \cdots
\end{array}$$

Also, if M is compact and R-oriented with fundamental class $[M]$ restricting along K to $[M]_K$, we have the ladder of isomorphisms

$$\begin{array}{ccccccccccc}
\cdots \to & \check{H}^p(M,L) & \to & \check{H}^p(K,L) & \to & \check{H}^{p+1}(M,K) & \to & \check{H}^{p+1}(M,L) & \to \cdots \\
& \downarrow{\scriptstyle \cap[M]} & & \downarrow{\scriptstyle \cap[M]_K} & & \downarrow{\scriptstyle \cap[M]} & & \downarrow{\scriptstyle \cap[M]} & \\
\cdots \to & H_q(M-L) & \to & H_q(M-L, M-K) & \to & H_{q-1}(M-K) & \to & H_{q-1}(M-L) & \to \cdots
\end{array}$$

To prove this theorem, we will follow the same five-step process we used to prove the Orientation Theorem 32.1. We have already prepared the Mayer-Vietoris ladder for this purpose. We will also need:

Lemma 37.2. *Let $A_1 \supseteq A_2 \supseteq \cdots$ be a decreasing sequence of compact subspaces of M. Then*

$$\check{H}^p(A_k) \to \check{H}^p(A)$$

is an isomorphism.

Proof. This follows from the observation that a direct limit of direct limits is a direct limit. □

Proof of Theorem 37.1. By the top ladder and the five-lemma, we may assume $L = \varnothing$; so we want to prove that

$$\cap[M]_K : \check{H}^p(K; R) \to H_q(M, M-K; R)$$

is an isomorphism.

(1) $M = \mathbf{R}^n$, K a compact convex set. We claim that

$$\check{H}^*(K) \xrightarrow{\cong} H^*(K)\,.$$

For any $\epsilon > 0$, let U_ϵ denote the ϵ-neighborhood of K,

$$U_\epsilon = \bigcup_{x \in K} B_\epsilon(x)\,.$$

For any $y \in U_\epsilon$ there is a closest point in K, since the distance function to y is continuous and bounded below on the compact set K and so achieves its infimum. If $x', x'' \in K$ are the same distance from y, then the midpoint of the segment joining x' and x'' is closer, but lies in K since K is convex. So there is a unique closest point, $f(y)$. We let the listener check that $f : U_\epsilon \to K$ is continuous. It is also clear that if $i : K \to U_\epsilon$ is the inclusion then $i \circ f$ is homotopic to the identity on Y, by an affine homotopy.

Now let D^n be a disk centered at the origin and containing the compact set K, and consider the commutative diagram

$$\begin{array}{ccc} H^p(K) & \xrightarrow{\cap[\mathbf{R}^n]_K} & H_q(\mathbf{R}^n, \mathbf{R}^n - K) \\ \uparrow \cong & & \uparrow \cong \\ H^p(D^n) & \longrightarrow & H_q(\mathbf{R}^n, \mathbf{R}^n - D^n) \\ \downarrow \cong & & \downarrow \cong \\ H^p(*) & \longrightarrow & H_q(\mathbf{R}^n, \mathbf{R}^n - *)\,. \end{array}$$

The groups are zero unless $p = 0, q = n$. By naturality of the cap product, the bottom map is given by $1 \mapsto 1 \cap [\mathbf{R}^n]_*$, and this is $[\mathbf{R}^n]_*$ since capping with 1 is the identity, and this fundamental class is a generator of $H_n(\mathbf{R}^n, \mathbf{R}^n - *)$.

(2) K a finite union of compact convex subsets of $\mathbf{R}^n$. This follows by induction and the five lemma applied to the Mayer-Vietoris ladder 36.2.

(3) K is any compact subset of $\mathbf{R}^n$. This follows as before by a limit argument, using Lemmas 32.4 and 37.2.

(4) M arbitrary, K is a finite union of compact Euclidean subsets of M. This follows from (3) and Theorem 36.2.

(5) M arbitrary, K an arbitrary compact subset. This follows just as in the proof of Theorem 32.1. □

Let's point out some special cases. With $K = M$, we get:

Corollary 37.3. *Suppose that M is a compact R-oriented n-manifold, and let L be a closed subset. Then (with $p + q = n$) we have the commuting ladder whose rungs are isomorphisms:*

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \check{H}^{p-1}(L) & \longrightarrow & \check{H}^p(M, L) & \longrightarrow & H^p(M) & \longrightarrow & \check{H}^p(L) & \longrightarrow & \cdots \\ & & \downarrow \cap[M]_L & & \downarrow \cap[M] & & \downarrow \cap[M] & & \downarrow \cap[M]_L & & \\ \cdots & \longrightarrow & H_{q+1}(M, M - L) & \longrightarrow & H_q(M - L) & \longrightarrow & H_q(M) & \longrightarrow & H_q(M, M - L) & \longrightarrow & \cdots \end{array}$$

With $L = \varnothing$, we get:

Corollary 37.4. *Suppose that M is an n-manifold, and let K be a compact subset. An R-orientation along K determines (with $p + q = n$) an isomorphism*

$$\cap[M]_K : \check{H}^p(K; R) \to H_q(M, M - K; R)\,.$$

The intersection of these two special cases is:

Corollary 37.5 (Poincaré duality)**.** *Let M be a compact R-oriented n-manifold. Then*

$$\cap[M] : H^p(M; R) \to H_{n-p}(M; R)$$

is an isomorphism.