18 Euler characteristic and homology approximation

Theorem 18.1. *Let X be a finite CW-complex with a_n n-cells. Then*

$$\chi(X) = \sum_{k=0}^{\infty}(-1)^k a_k$$

depends only on the homotopy type of X; it is independent of the choice of CW structure.

This integer $\chi(X)$ is called the *Euler characteristic* of X. We will prove this theorem by showing that $\chi(X)$ equals a number computed from the homology groups of X, which are themselves homotopy invariants.

We'll need a little bit of information about the structure of finitely generated abelian groups.

Let A be an abelian group. The set of *torsion* elements of A,

$$\mathrm{Tors}(A) = \{a \in A : na = 0 \text{ for some } n \neq 0\}\,,$$

is a subgroup of A. A group is *torsion free* if $\mathrm{Tors}(A) = 0$. For any A the quotient group $A/\mathrm{Tors}(A)$ is torsion free.

For a general abelian group, that's about all you can say. But now assume A is finitely generated. Then $\mathrm{Tors}(A)$ is a finite abelian group and $A/\mathrm{Tors}(A)$ is a finitely generated free abelian group, isomorphic to $\mathbf{Z}^r$ for some integer r called the *rank* of A. Pick elements of A that map to a set of generators of $A/\mathrm{Tors}(A)$, and use them to define a map $A/\mathrm{Tors}A \to A$ splitting the projection map. This shows that if A is finitely generated then

$$A \cong \mathrm{Tors}(A) \oplus \mathbf{Z}^r\,.$$

A finite abelian group A is necessarily of the form

$$\mathbf{Z}/n_1 \oplus \mathbf{Z}/n_2 \oplus \cdots \oplus \mathbf{Z}/n_t \text{ where } n_1|n_2|\cdots|n_t\,.$$

The n_i are the "torsion coefficients" of A. They are well defined natural numbers.

Lemma 18.2. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of finitely generated abelian groups. Then*

$$\mathrm{rank}\,A - \mathrm{rank}\,B + \mathrm{rank}\,C = 0\,.$$

Theorem 18.3. *Let X be a finite CW complex. Then*

$$\chi(X) = \sum_k (-1)^k \mathrm{rank}\,H_k(X)\,.$$

Proof. Pick a CW-structure with, say, a_k k-cells for each k. We have the cellular chain complex C_*. Write H_*, Z_*, and B_* for the homology, the cycles, and the boundaries, in this chain complex. From the definitions, we have two families of short exact sequences:

$$0 \to Z_k \to C_k \to B_{k-1} \to 0$$

and

$$0 \to B_k \to Z_k \to H_k \to 0\,.$$

Let's use them and facts about rank rewrite the alternating sum:

$$\begin{aligned}\sum_k (-1)^k a_k &= \sum_k (-1)^k \mathrm{rank}(C_k)\\ &= \sum_k (-1)^k (\mathrm{rank}\,(Z_k) + \mathrm{rank}\,(B_{k-1}))\\ &= \sum_k (-1)^k (\mathrm{rank}\,(B_k) + \mathrm{rank}\,(H_k) + \mathrm{rank}\,(B_{k-1}))\end{aligned}$$

The terms $\mathrm{rank}\,B_k + \mathrm{rank}\,B_{k-1}$ cancel because it's an alternating sum. This leaves $\sum_k (-1)^k \mathrm{rank}\,H_k$. But $H_k \cong H_k^{\mathrm{sing}}(X)$. □

In the early part of the 20th century, "homology groups" were not discussed. It was Emmy Noether who first described things that way. Instead, people worked mainly with the sequence of ranks,

$$\beta_k = \operatorname{rank} H_k(X)\,,$$

which are known (following Poincaré) as the *Betti numbers* of X.

Given a CW-complex X of finite type, can we give a lower bound on the number of k-cells in terms of the homology of X? Let's see. $H_k(X)$ is finitely generated because $C_k(X) \hookleftarrow Z_k(X) \twoheadrightarrow H_k(X)$. Thus

$$H_k(X) = \bigoplus_{i=1}^{t(k)} \mathbf{Z}/n_i(k)\mathbf{Z} \oplus \mathbf{Z}^{r(k)}$$

where the $n_1(k)|\cdots|n_{t(k)}(k)$ are the torsion coefficients of $H_k(X)$ and $r(k)$ is the rank.

The minimal chain complex with $H_k = \mathbf{Z}^r$ and $H_q = 0$ for $q \neq k$ is just the chain complex with 0 everywhere except for $\mathbf{Z}^r$ in the kth degree. The minimal chain complex *of free abelian groups* with $H_k = \mathbf{Z}/n\mathbf{Z}$ and $H_q = 0$ for $q \neq k$ is the chain complex with 0 everywhere except in dimensions $k+1$ and k, where we have $\mathbf{Z} \xrightarrow{n} \mathbf{Z}$ These small complexes are called *elementary chain complexes.*

This implies that a lower bound on the number of k-cells is

$$r(k) + t(k) + t(k-1)\,.$$

The first two terms give generators for H_k, and the last gives relations for H_{k-1}.

These elementary chain complexes can be realized as the reduced cellular chains of CW complexes (at least if $k > 0$). A wedge of r copies of S^k has a CW structure with one 0-cell and r k-cells, so its cellular chain complex has $\mathbf{Z}^r$ in dimension k and 0 in other positive dimensions. To construct a CW complex with cellular chain complex given by $\mathbf{Z} \xrightarrow{n} \mathbf{Z}$ in dimensions $k+1$ and k and 0 in other positive dimensions, start with S^k as k-skeleton and attach a $k+1$-cell by a map of degree n. For example, when $k = 1$ and $n = 2$, you have $\mathbf{RP}^2$. These CW complexes are called "Moore spaces."

This maximally efficient construction of a CW complex in a homotopy type can in fact be achieved, at least in the simply connected case:

Theorem 18.4 (Wall, [10])**.** *Let X be a simply connected CW-complex of finite type. Then there exists a CW complex Y with $r(k) + t(k) + t(k-1)$ k-cells, for all k, and a homotopy equivalence $Y \to X$.*

We will prove this theorem in 18.906.

The construction of Moore spaces can be generalized:

Proposition 18.5. *For any graded abelian group A_* with $A_k = 0$ for $k \leq 0$, there exists a CW complex X with $\widetilde{H}_*(X) = A_*$.*

Proof. Let A be any abelian group. Pick generators for A. They determine a surjection from a free abelian group F_0. The kernel F_1 of that surjection is free, being a subgroup of a free abelian group. Write G_0 for minimal set of generators of F_0, and G_1 for a minimal set of generators for F_1.

Let $k \geq 1$. Define X_k to be the wedge of $|G_0|$ copies of S^k, so $H_k(X_k) = \mathbf{Z}G_0$. Now define an attaching map

$$\alpha : \coprod_{b \in G_1} S^k_b \to X_k$$

by specifying it on each summand S^k_b. The generator $b \in G_1$ is given by a linear combination of the generators of F_0, say

$$b = \sum_{i=1}^{s} n_i a_i \,.$$

We want to mimic this in topology. To do this, first map $S^k \to \bigvee^s S^k$ by pinching $(s-1)$ tangent circles to points. In homology, this map takes a generator of $H_k(S^k)$ to the sum of the generators of the k-dimensional homology of the various spheres in the bouquet. Map the ith sphere in the wedge to $S^k_{a_i} \subseteq X_k$ by a map of degree n_i. The map on the summand S^k_b is then the composite of these two maps,

$$S^k_b \to \bigvee_{i=1}^{s} S^k \to \bigvee_a S^k_a \,.$$

Altogether, we get a map α that realizes $F_1 \to F_0$ in H_k. So using it as an attaching map produces a CW complex X with $\widetilde{H}_q(X) = A$ for $q = k$ and 0 otherwise. Write $M(A,k)$ for a CW complex produced in this way.

Finally, given a graded abelian group A_*, form the wedge over k of the spaces $M(A_k, k)$. □

Such a space $M(A,k)$, with $\widetilde{H}_q(M(A,k)) = A$ for $q = k$ and 0 otherwise, is called a *Moore space of type* (A,k) [9]. The notation is a bit deceptive, since $M(A,k)$ cannot be made into a functor **Ab** $\to$ Ho**Top**.

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