12 Subdivision

We will begin the proof of the locality principle today, and finish it in the next lecture. The key is a process of subdivision of singular simplices. It will use the "cone construction" $b*$ from Lecture 5. The cone construction dealt with a region X in Euclidean space, star-shaped with respect to $b \in X$, and gave a chain-homotopy between the identity and the "constant map" on $S_*(X)$:

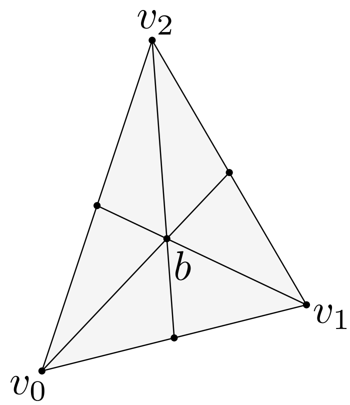
$$db * + b * d = 1 - \eta\epsilon$$

where $\epsilon : S_*(X) \to \mathbf{Z}$ is the augmentation and $\eta : \mathbf{Z} \to S_*(X)$ sends 1 to the constant 0-chain c_b^0.

Let's see how the cone construction can be used to "subdivide" an "affine simplex." An *affine simplex* is the convex hull of a finite set of points in Euclidean space. To make this non-degenerate, assume that the points $v_0, v_1, \ldots, v_n$, have the property that $\{v_1 - v_0, \ldots, v_n - b_0\}$ is linearly independent. The *barycenter* of this simplex is the center of mass of the vertices,

$$b = \frac{1}{n+1} \sum v_i \,.$$

Start with $n = 1$. To subdivide a 1-simplex, just cut it in half. For the 2-simplex, look at the subdivision of each face, and form the cone of them with the barycenter of the 2-simplex. This gives us a decomposition of the 2-simplex into six sub-simplices.



We want to formalize this process, and extend it to singular simplices (using naturality, of course). Define a natural transformation

$$\$: S_n(X) \to S_n(X)$$

by defining it on standard n-simplex, namely by specifying what $\$(\iota_n)$ is where $\iota_n : \Delta^n \to \Delta^n$ is the universal n-simplex, and then extending by naturality:

$$\$(\sigma) = \sigma_* \$(\iota_n)\,.$$

Here's the definition. When $n = 0$, define \$ to be the identity; i.e., $\$\iota_0 = \iota_0$. For $n > 0$, define

$$\$\iota_n := b_n * \$d\iota_n$$

where b_n is the barycenter of Δ^n. This makes a lot of sense if you draw out a picture, and it's a very clever definition that captures the geometry we described.

The dollar sign symbol is a little odd, but consider: it derives from the symbol for the Spanish piece of eight, which was meant to be subdivided (so for example two bits is a quarter).

Here's what we'll prove.

Proposition 12.1. \$ *is a natural chain map* $S_*(X) \to S_*(X)$ *that is naturally chain-homotopic to the identity.*

Proof. Let's begin by proving that it's a chain map. We'll use induction on n. It's enough to show that $d\$\iota_n = \$d\iota_n$, because then, for any n-simplex σ,

$$d\$\sigma = d\$\sigma_*\iota_n = \sigma_* d\$\iota_n = \sigma_*\$d\iota_n = \$d\sigma_*\iota_n = \$d\sigma\,.$$

Dimension zero is easy: since $S_{-1} = 0$, $d\$\iota_0$ and $\$d\iota_0$ are both zero and hence equal.

For $n \geq 1$, we want to compute $d\$\iota_n$. This is:

$$\begin{aligned} d\$\iota_n &= d(b_n * \$d\iota_n) \\ &= (1 - \eta_b\epsilon - b_n * d)(\$d\iota_n) \end{aligned}$$

What happens when $n = 1$? Well,

$$\eta_b\epsilon\$d\iota_1 = \eta_b\epsilon\$(c_1^0 - c_0^0) = \eta_b\epsilon(c_1^0 - c_0^0) = 0\,,$$

since ϵ takes sums of coefficients. So the $\eta_b\epsilon$ term drops out for any $n \geq 1$. Let's continue, using the inductive hypothesis:

$$\begin{aligned} d\$\iota_n &= (1 - b_n * d)(\$d\iota_n) \\ &= \$d\iota_n - b_n * d\$d\iota_n \\ &= \$d\iota_n - b_n\$d^2\iota_n \\ &= \$d\iota_n \end{aligned}$$

because $d^2 = 0$.

To define the chain homotopy T, we'll just write down a formula and not try to justify it. Making use of naturality, we just need to define $T\iota_n$. Here it is:

$$T\iota_n = b_n * (\$\iota_n - \iota_n - Td\iota_n) \in S_{n+1}(\Delta^n)\,.$$

Once again, we're going to check that T is a chain homotopy by induction, and, again, we need to check only on the universal case.

When $n = 0$, the formula gives $T\iota_0 = 0$ (which starts the inductive definition!) so it's true that $dT\iota_0 - Td\iota_0 = \$\iota_0 - \iota_0$. Now let's assume that $dTc - Tdc = \$c - c$ for every $(n-1)$-chain c. Let's start by computing $dT\iota_n$:

$$\begin{aligned} dT\iota_n &= d_n(b_n * (\$\iota_n - \iota_n - Td\iota_n)) \\ &= (1 - b_n * d)(\$\iota_n - \iota_n - Td\iota_n) \\ &= \$\iota_n - \iota_n - Td\iota_n - b_n * (d\$\iota_n - d\iota_n - dTd\iota_n) \end{aligned}$$

All we want now is that $b_n * (d\$\iota_n - d\iota_n - dTd\iota_n) = 0$. We can do this using the inductive hypothesis, because $d\iota_n$ is in dimension $n - 1$.

$$\begin{aligned} dTd\iota_n &= -Td(d\iota_n) + \$d\iota_n - d\iota_n \\ &= \$d\iota_n - d\iota_n \\ &= d\$\iota_n - d\iota_n\,. \end{aligned}$$

This means that $d\$\iota_n - d\iota_n - dT\,d\iota_n = 0$, so T is indeed a chain homotopy. □