Introduction to Orbifolds

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1 Introduction

Orbifolds lie at the intersection of many different areas of mathematics, including algebraic and differential geometry, topology, algebra and string theory. Orbifolds were first introduced into topology and differential geometry by Satake [6], who called them *V-manifolds.* Satake described them as topological spaces generalizing smooth manifolds and generalized concepts such as de Rham cohomology and the Gauss-Bonnet theorem to orbifolds.

In the late 1970s, orbifolds were use by Thurston in his geometrization program for three-manifolds. It was Thurston who changed the name from *V-manifold* to orbifold. In 1985, with the work of Dixon, Harvey, Vafa and Witten on conformal field theory [7], the interest on orbifolds dramatically increased, due to their role in string theory, even though orbifolds were already very important objects in mathematics.

As Thurston mentions in [1, p. 297], it is often more effective to study the quotient manifold of a group acting freely and properly discontinuously on a space rather than to limit one's image to the group action alone. In the same spirit, it is often more effective to study the quotient spaces of groups acting properly discontinuously, but not necessarily freely, on a topological space rather than to limit one's image to the action alone. Since a way to construct orbifolds is by taking the quotient of a manifold by some properly discontinuous group action, as we will see in the next sections, the study of orbifolds often simplify the analysis of more complicated structures, such as three-manifolds, for example.

From the ideas discussed in the paragraph above, we can think of orbifolds as a space with isolated singularities, that is, a space that looks like a quotient manifold of a group acting on a space, together with some additional information about the action of the group on points where the action is not free. For instance, if I consider the quotient space of the disk D^2 by the action of the group of rotations of order 3 around the center of the disk, our orbifold would be the quotient space together with information telling me that the group of rotations acts as the cyclic group of order 3 at the origin.

In this paper, we introduce the basics of the topology of orbifolds, talk about their fundamental groups and state an orbifold version of van Kampen's theorem. With this machinery, we show that $PSL_2(\mathbb{Z})$ is isomorphic to the free product of a cyclic group of order two and another of order three.

2 Transformation Groups

Throughout this paper, all topological groups are assumed to be Hausdorff.

Definition 2.1. An *action* of a topological group G on a space X is a continuous map $G \times X \to X$, denoted by $(g, x) \mapsto gx$, so that $g(hx) = (gh)x$ and that $1x = x$.

Given $x \in X$, we define the group $G_x = \{g \in G \mid gx = x\}$ as the *isotropy subgroup* of x. The isotropy subgroup G_x of any point x is a closed subgroup of G. The action is said to be *free* if $G_x = \{1\}$, for all $x \in X$. The set $G(x) = \{gx \in X \mid g \in G\}$ is the *orbit* of x. The action is said to be *transitive* if $G(x) = X$.

Given $x \in X$, the natural map $\lambda : G/G_x \to G(x)$ defined by $gG_X \mapsto gx$ is a continuous bijection. The *orbit space* X/G is the set of orbits in X endowed with the quotient topology (with respect to the natural map $X \to X/G$, which we will call the *orbit map*).

Let X be Hausdorff, on which G acts continuously and transitively. If we fix a point $x \in X$, then for any $U \subset X$, we have that $\lambda^{-1}(U) = \pi(\{g \in G \mid gx \in U\})$, where $\pi : G \to G/G_x$ is the projection map. This equality implies that $\lambda^{-1}(U)$ is open if U is open, which implies that λ is continuous. But λ is not necessarily a homeomorphism. However, if G and X are locally compact and if G has a countable basis of open sets then this map λ is a homeomorphism, see [3, p. 2]. Therefore, we can state the following theorem:

Theorem 2.2. The map $\lambda : G/G_x \to X$ is a homeomorphism if both G and X are locally compact and if G has a countable basis of open sets.

If Γ is a discrete group and X is a Hausdorff space such that Γ acts on X, this action is said to be *properly discontinuous,* if given two points $x, y \in X$, there are open neighborhoods U of x and V of y for which $(\gamma U) \cap V \neq \emptyset$ for only finitely many $\gamma \in \Gamma$.

3 Orbifolds

3.1 Definition of Orbifolds

Definition 3.1. An n-dimensional *orbifold chart* on a topological space X is a 3-tuple $(\tilde{U}, G, \pi)$, where

- $\tilde{U}$ is open in $\mathbb{R}^n$,
- G is a finite group of homeomorphisms of $\tilde{U}$,
- $\pi : \tilde{U} \to X$ is a map defined by $\pi = \overline{\pi} \circ p$, where $p : \tilde{U} \to \tilde{U}/G$ is the orbit map and $\overline{\pi} : \tilde{U}/G \to X$ is a map that induces a homeomorphism of $\tilde{U}/G$ onto an open subset $U \subset X$.

An *embedding* $\lambda : (\tilde{U}_1, G_1, \pi_1) \to (\tilde{U}_2, G_2, \pi_2)$ between two charts is a smooth embedding $\lambda : \tilde{U}_1 \to \tilde{U}_2$ such that $\pi_2 \circ \lambda = \pi_1$.

For $i = 1, 2$, let $(\tilde{U}_i, G_i, \pi_i)$ be two orbifold charts on X such that $U_i = \pi_i(\tilde{U}_i)$, and x is a point in $U_1 \cap U_2$. We say that these charts are *compatible* if there exists an open neighborhood $V \subset U_1 \cap U_2$ of x and a chart $(\tilde{V}, H, \phi)$ with $\phi(\tilde{V}) = V$ such that there are two embeddings $\lambda_i : (\tilde{V}, H, \phi) \to (\tilde{U}_i, G_i, \pi_i)$.

Definition 3.2. An n-dimensional *orbifold atlas* on X is a collection $\mathcal{U} = \{(\tilde{U}_\alpha, G_\alpha, \pi_\alpha)\}_{\alpha \in I}$ of compatible n-dimensional orbifold charts which cover X.

Definition 3.3. An *orbifold* $\mathcal{O}$ of dimension n consists of a paracompact Hausdorff space $X_{\mathcal{O}}$ together with an n-dimensional orbifold atlas of charts $\mathcal{U}_{\mathcal{O}}$.

Example 3.4. A manifold is an orbifold where each G_α is the trivial group, so that we get $\tilde{U}_\alpha$ homeomorphic to U_α.

Definition 3.5. Let $\mathcal{O}_1 = (X_{\mathcal{O}_1}, \mathcal{U}_{\mathcal{O}_1})$ and $\mathcal{O}_2 = (X_{\mathcal{O}_2}, \mathcal{U}_{\mathcal{O}_2})$ be two orbifolds. A map $f : X_{\mathcal{O}_1} \to X_{\mathcal{O}_2}$ is a *smooth map between orbifolds* if for any point $x \in X$ there are charts $(\tilde{U}_1, G_1, \pi_1)$ around x and $(\tilde{U}_2, G_2, \pi_2)$ around $f(x)$ such that f maps $\pi_1(\tilde{U}_1)$ into $\pi_2(\tilde{U}_2)$ and can be lifted to a smooth map $\tilde{f} : \tilde{U}_1 \to \tilde{U}_2$ such that $\pi_2 \tilde{f} = f \pi_1$.

To be able to state van Kampen's theorem in the orbifold version, we also need to define a *suborbifold.* Therefore, we make the following definition:

Definition 3.6. A *suborbifold* $\mathcal{O}_1$ of an orbifold $\mathcal{O}$ is a subspace $X_{\mathcal{O}_1} \subset X_{\mathcal{O}}$ together with an embedding $\iota : X_{\mathcal{O}_1} \hookrightarrow X_{\mathcal{O}}$.

3.2 Quotient Orbifolds

Proposition 3.7. If M is a manifold and G is a group acting properly discontinuously on M, then M/G has the structure of an orbifold.

Proof. For any point $x \in M/G$, choose $\tilde{x} \in M$ projecting to x. Let I_x be the isotropy group of $\tilde{x}$ (I_x depends of course on the particular choice of $\tilde{x}$). There is a neighborhood $\tilde{U}_x$ of $\tilde{x}$ invariant by I_x and disjoint from its translates by elements of G not in I_x. The projection of $U_x = \tilde{U}_x / I_x$ is a homeomorphism. To obtain a suitable cover of M/G, augment some cover $\{U_x\}$ by adjoining finite intersections. Whenever $U_{x_1} \cap \ldots \cap U_{x_k} \neq \emptyset$, this means that some set of translates $g_1 \tilde{U}_{x_1} \cap \ldots \cap g_k \tilde{U}_{x_k}$ has a corresponding non-empty intersection. This intersection may be taken to be

$$\widetilde{\overbrace{U_{x_1} \cap \ldots \cap U_{x_k}}}$$

with associated group $g_1 I_{x_1} g_1^{-1} \cap \ldots \cap g_k I_{x_k} g_k^{-1}$ acting on it. $\square$

Observation. From now on, we are going to use the notation $[M/G]$ to mean M/G as an orbifold.

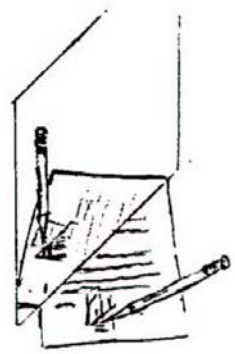
Note that each point x in an orbifold $\mathcal{O}$ has an associated group G_x, which is well defined up to conjugation: in a local coordinate system, $U = \tilde{U}/G$, G_x is the isotropy group of any point in $\tilde{U}$ corresponding to x.

The set

$$\Sigma_{\mathcal{O}} = \{x \in X \mid G_x \neq \{1\}\}$$

is the *singular locus of* $\mathcal{O}$.

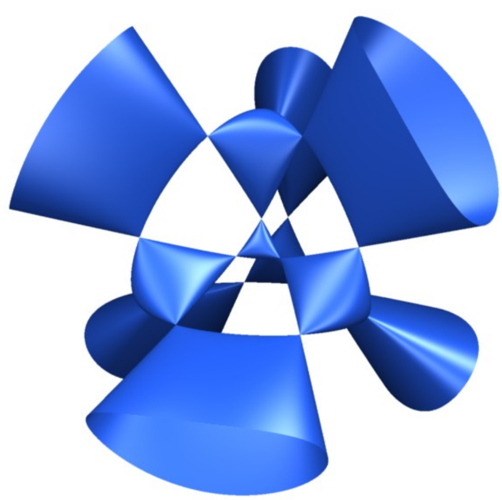
Example 3.8. Consider the action of $\mathbb{Z}_2$ on $\mathbb{R}^3$ by reflection in the $y - z$ plane. The quotient space $\mathbb{R}^3/\mathbb{Z}_2$ is the half-space $x \geq 0$, which we will denote by $\mathcal{F}$. Physically, one may imagine a mirror placed on the $y - z$ wall of $\mathcal{F}$, as shown below. This quotient space has an orbifold structure where each point z on the boundary of $\mathcal{F}$ has a neighborhood homeomorphic to the quotient of a neighborhood $U \subset \mathbb{R}^3$ of z modulo $\mathbb{Z}_2$. If z is a point of $\mathcal{F}$ not on the boundary, then there exists a neighborhood $U \subset \mathbb{R}^3$ on which $\mathbb{Z}_2$ acts freely. Therefore, this neighborhood is canonically homeomorphic to itself.



Example 3.9. One example which shows how orbifolds appear in algebraic geometry is given by the *Kummer surface,* which is defined by the action of $\mathbb{Z}_2$ on $\mathbb{T}^4$ defined by

$$\sigma(e^{it_1}, e^{it_2}, e^{it_3}, e^{it_4}) = (e^{\ \ it_1}, e^{\ \ it_2}, e^{\ \ it_3}, e^{\ \ it_4})$$

where σ corresponds to the matrix $\ \ I$. The orbifold $[\mathbb{T}^4/\mathbb{Z}_2]$ has sixteen isolated singular points. You can see it in the picture below.



3.3 Fundamental Groups of Orbifolds

When M in Proposition 3.7 is simply connected, then M plays the role of universal covering space and G plays the role of the fundamental group of $[M/G]$. To formalize this, we will now define the notion of a covering orbifold.

Definition 3.10. A *covering orbifold* of an orbifold $\mathcal{O}$ is an orbifold $\tilde{\mathcal{O}}$ with a projection $p : X_{\tilde{\mathcal{O}}} \to X_{\mathcal{O}}$ between the underlying spaces, such that the following conditions hold:

- each point $x \in X_{\mathcal{O}}$ has a neighborhood $U = \pi(\tilde{U})$ (where $(\tilde{U}, G, \pi)$ is a chart of $\mathcal{O}$), for which each component V_i of $p^{-1}(U)$ is homeomorphic to $\tilde{U}/G_i$, where G_i is some subgroup of G,
- if $\overline{\psi}_i : \tilde{U}/G_i \to V_i$ is the homeomorphism above, $\pi_i : \tilde{U} \to \tilde{U}/G_i$ is the quotient map and $\psi_i = \overline{\psi}_i \circ \pi_i$, we must have $\pi = p \circ \psi_i$.

Observation. The underlying space $X_{\tilde{\mathcal{O}}}$ is not generally a covering space of $X_{\mathcal{O}}$.

Proposition 3.11 ([1] p. 305)**.** Any orbifold $\mathcal{O}$ has a universal cover $\tilde{\mathcal{O}}$ with a projection $p : \tilde{\mathcal{O}} \to \mathcal{O}$. In other words, if $x \in X_{\mathcal{O}} - \Sigma_{\mathcal{O}}$ is a base point for $\mathcal{O}$, then

$$\tilde{\mathcal{O}} \xrightarrow{p} \mathcal{O}$$

is a connected covering orbifold with base point $\tilde{x}$ which projects to x, such that for any other covering orbifold

$$\mathcal{O}' \xrightarrow{p'} \mathcal{O}$$

with base point x', such that $p'(x') = x$, there is a lifting $q : \tilde{\mathcal{O}} \to \mathcal{O}'$ of p to a covering map of $\mathcal{O}'$.

The universal cover $\tilde{\mathcal{O}}$ of an orbifold $\mathcal{O}$ is automatically a normal cover ([5, p. 70, 71]): for any preimage of $\tilde{x}$ of the base point x there is a deck transformation taking $\tilde{x}$ to x.

Definition 3.12. The fundamental group of an orbifold $\mathcal{O}$ is the group of deck transformations of the universal cover $\tilde{\mathcal{O}}$.

This definition justifies the remark in the beginning of this section.

Example 3.13. Consider the action of $\mathbb{Z}_3$ on the complex plane $\mathbb{C}$ by rotations of order 3 around the origin. The quotient space is given by

$$\mathbb{C}/\mathbb{Z}_3 \approx \{z \in \mathbb{C} \mid z = |z|e^{i\theta},\ 0 \leq \theta < 2\pi/3\}.$$

Since the complex plane is a simply connected manifold and $\mathbb{Z}_3$ acts properly discontinuously on $\mathbb{C}$, we have that $\mathbb{C}/\mathbb{Z}_3$ has the structure of an orbifold and $\mathbb{C}$, together with the quotient map, is a universal cover. By Proposition 1.40 in [5, p. 72], $\mathbb{Z}_3$ is the group of deck transformations of this covering space and therefore we have that $\pi_1^{\mathrm{orb}}([\mathbb{C}/\mathbb{Z}_3]) = \mathbb{Z}_3$.

The fundamental groups of orbifolds can be computed in much the same ways as the fundamental groups of manifolds [1, p. 307]. In particular, we have a version of van Kampen's theorem for orbifolds, as stated below:

Theorem 3.14 (van Kampen for orbifolds)**.** If $\mathcal{O}, \mathcal{O}_1$ and $\mathcal{O}_2$ are orbifolds such that $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$, where $\mathcal{O}_1 \cap \mathcal{O}_2$ is path connected, then:

$$\pi_1^{\text{orb}}(\mathcal{O}) \approx \pi_1^{\text{orb}}(\mathcal{O}_1) *_{\pi_1^{\text{orb}}(\mathcal{O}_1 \cap \mathcal{O}_2)} \pi_1^{\text{orb}}(\mathcal{O}_2).$$

Corollary 3.15. If $\mathcal{O}, \mathcal{O}_1$ and $\mathcal{O}_2$ are orbifolds such that $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$, where $\mathcal{O}_1 \cap \mathcal{O}_2$ is path connected, and if $\mathcal{O}_2$ is a simply connected manifold, then $\pi_1^{\text{orb}}(\mathcal{O}) \approx \pi_1^{\text{orb}}(\mathcal{O}_1)$.

In particular, the corollary above states that if we have an orbifold $\mathcal{O}$ whose underlying space is simply connected and which has only one singular point x, then the fundamental group of $\mathcal{O}$ is the isotropy group of x.

4 Projective Special Linear Group

The projective special linear group $\text{PSL}(2, \mathbb{Z})$ is the quotient of $\text{SL}(2, \mathbb{Z})$ by its center $\{I, -I\}$, with group operation being multiplication of matrices. The projective special linear group is isomorphic to the modular group Γ, that is, the group of linear fractional transformations of the upper half of the complex plane (we will call the upper half plane $\mathcal{H}$ from now on) which have the form

$$z \mapsto \frac{az + b}{cz + d}$$

where $a,\ b,\ c$, and d are integers such that $ad - bc = 1$. The group operation is function composition.

Let us now consider the action of $\text{SL}(2, \mathbb{R})$ on $\mathcal{H}$ defined by $\sigma z = \dfrac{az + b}{cz + d}$ where $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an element of $\text{SL}(2, \mathbb{R})$ and $z \in \mathcal{H}$. This action is transitive, since for $a > 0$, $\begin{pmatrix} a^{\frac{1}{2}} & ba^{\frac{1}{2}} \\ 0 & a^{-\frac{1}{2}} \end{pmatrix}$ sends i to $ai + b$.

Let G_i be the isotropy group of $i \in \mathcal{H}$. It is easy to see that $G_i = \text{SO}(2)$. Therefore, by Theorem 2.2, $\mathcal{H}$ is homeomorphic to $\text{SL}(2, \mathbb{R})/G_i = \text{SL}(2, \mathbb{R})/\text{SO}(2)$ through the map $\lambda : \text{SL}(2, \mathbb{R})/\text{SO}(2) \to \mathcal{H}$ defined by $\lambda(\gamma) = \gamma i$.

Since $\mathcal{H}$ is homeomorphic to $\text{SL}(2, \mathbb{R})/\text{SO}(2)$, and since $\text{PSL}(2, \mathbb{Z})$ is a discrete subgroup of $\text{SL}(2, \mathbb{R})$, we have that $\text{PSL}(2, \mathbb{Z})$ acts properly discontinuously on $\mathcal{H}$, due to Proposition 1.6 in [3, p. 3].

Now that we know that $\text{PSL}(2, \mathbb{Z})$ acts properly discontinuously on $\mathcal{H}$, we will find the fundamental domain for $\mathcal{H}/\text{PSL}(2, \mathbb{Z})$.

Definition 4.1. For any discrete subgroup G of $\text{SL}(2, \mathbb{R})$, we call F a *fundamental domain* for $\mathcal{H}/G$ if F satisfies the following conditions:

(i) F is a connected subset of $\mathcal{H}$

(ii) no two points of F are equivalent under G

(iii) every point of $\mathcal{H}$ is equivalent to some point of the closure of F under G

Proposition 4.2. A fundamental domain F for $\mathcal{H}/\mathrm{PSL}(2,\mathbb{Z})$ is given by:

$$F = \{w \in \mathbb{C} \mid -\frac{1}{2} < Re(w) < \frac{1}{2}, |w| > 1\}.$$

Proof. Let $z \in H$ and $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z})$. Then

$$\mathrm{Im}(\sigma z) = \mathrm{Im}(\frac{az+b}{cz+d}) = \mathrm{Im}(\frac{az+b}{cz+d} \cdot \frac{c\bar{z}+d}{c\bar{z}+d}) = \frac{(ad-bc)\mathrm{Im}(z)}{|cz+d|^2} = \frac{\mathrm{Im}(z)}{|cz+d|^2}.$$

Since $\{cz+d \mid (c,d) \in \mathbb{Z}^2\}$ is a lattice in $\mathbb{C}$, we have that $\min(|cz+d|)$ exists, for $(c,d) \neq (0,0)$. Thus, for a given z, this implies that $\max_{\sigma \in PSL(2,\mathbb{Z})} \{\mathrm{Im}(\sigma z)\}$ exists.

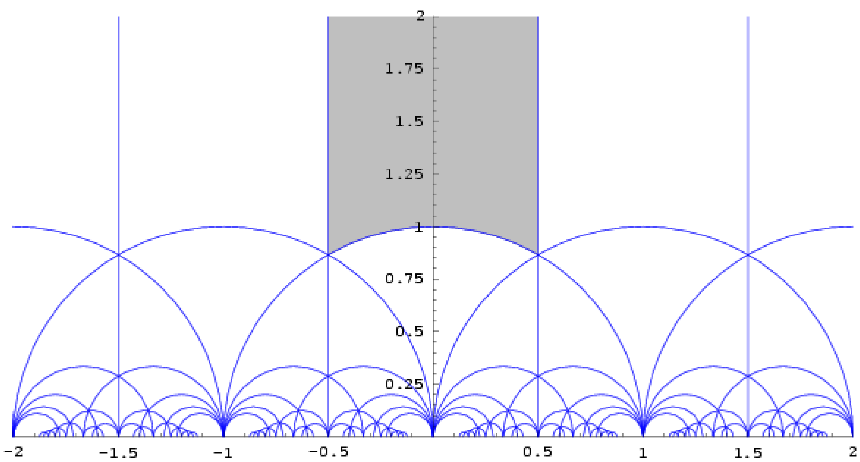
Let $\gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. If we have σ such that $\mathrm{Im}(\sigma z)$ is maximum, and if we set $w = \sigma z = x + iy$, where $x, y \in \mathbb{R}$, then

$$\mathrm{Im}(\gamma\sigma(z)) = \mathrm{Im}(-1/w) = \frac{y}{|w|^2} \leq y \Rightarrow |w| \geq 1$$

Thus, if $\tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, then $\mathrm{Im}(\tau^k\sigma(z)) = \mathrm{Im}(\sigma(z))$ for every $k \in \mathbb{Z}$, which implies that $|\tau^k\sigma(z)| \geq 1$. By a suitable choice of k, z is equivalent to a point of the region

$$\{w \in \mathbb{C} \mid -\frac{1}{2} \leq Re(w) \leq \frac{1}{2}, |w| \geq 1\}.$$

This region is the gray region shown below with its boundary. Denote by F the interior of the set above.



Now, to show that this F is a fundamental domain for $\mathrm{PSL}(2,\mathbb{Z})$, we only need to show that no two points in F are equivalent under the action of $\mathrm{PSL}(2,\mathbb{Z})$. We will prove this by contradiction.

Let z and z' be distinct points of F. Suppose that there exists a $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{PSL}(2, \mathbb{Z})$ such that $z' = \sigma z$. We can assume that $\mathrm{Im}(z) \leq \mathrm{Im}(z') = \mathrm{Im}(z)/|cz+d|^2$. Then, we have that

$$|c| \cdot \mathrm{Im}(z) = |c \cdot \mathrm{Im}(z) + 0| \leq |c \cdot \mathrm{Im}(z) + c \cdot \mathrm{Re}(z) + d| \leq |cz + d| \leq 1 \quad (*)$$

If $c = 0$, then $a = d = \pm 1$, hence $z' = z \pm b$, which is impossible. Therefore, we must have $c \neq 0$. Because $z \in F$, we must have $\mathrm{Im}(z) > \frac{\sqrt{3}}{2}$, which together with $(*)$, implies

$$|c| \cdot \frac{\sqrt{3}}{2} \leq |c| \cdot \mathrm{Im}(z) \leq 1 \Rightarrow |c| \leq \frac{2}{\sqrt{3}} \Rightarrow |c| = 1$$

Then, from $(*)$, we obtain $|z + d| \leq 1$. But if $z \in F$ and $|d| \geq 1$, we have $|z + d| > 1$. Therefore, we must have $d = 0$. But this implies that $|z| \leq 1$, which contradicts the fact that $z \in F$. $\square$

Since the transformation $z \mapsto z - 1$ takes any element $z \in \{w \in \mathcal{H} \mid \mathrm{Re}(w) = \frac{1}{2},\ |w| \geq 1\}$ to an element $z' \in \{w \in \mathcal{H} \mid \mathrm{Re}(w) = -\frac{1}{2},\ |w| \geq 1\}$ and since the transformation $z \mapsto -\frac{1}{z}$ takes any element $u \in \{w \in \mathcal{H} \mid 0 \leq \mathrm{Re}(w) \leq \frac{1}{2}, |w| = 1\}$ to an element $u' \in \{w \in \mathcal{H} \mid -\frac{1}{2} \leq \mathrm{Re}(w) \leq 0, |w| = 1\}$ we have that the set

$$F' = F \cup \{w \in \mathcal{H} \mid -\frac{1}{2} \leq \mathrm{Re}(w) \leq 0, |w| = 1\} \cup \{w \in \mathcal{H} \mid \mathrm{Re}(w) = -\frac{1}{2},\ |w| \geq 1\}$$

is a set of representatives for $\mathcal{H}$ modulo $\mathrm{PSL}(2, \mathbb{Z})$.

5 Application: Proof That $\mathrm{PSL}(2, \mathbb{Z}) \approx C_2 * C_3$

Theorem 5.1. The map

$$h : \mathbb{Z}/2 * \mathbb{Z}/3 \to \mathrm{PSL}(2, \mathbb{Z})$$

which takes $1_{\mathbb{Z}/2} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $1_{\mathbb{Z}/3} \mapsto \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ is an isomorphism.

Proof. In Section 4, we saw that $\mathrm{PSL}(2, \mathbb{Z})$ acts properly discontinuously on $\mathcal{H}$. Hence, by Proposition 3.7, we have that $\mathcal{H}/\mathrm{PSL}(2, \mathbb{Z})$ has the structure of an orbifold. Because $\mathcal{H}$ is simply connected, $\mathrm{PSL}(2, \mathbb{Z})$ is the fundamental group of the orbifold $[\mathcal{H}/\mathrm{PSL}(2, \mathbb{Z})]$, as we remarked in the beginning of Section 3.3.

Hence, to prove our claim, we only need to find the fundamental group of the orbifold $[\mathcal{H}/\mathrm{PSL}(2, \mathbb{Z})]$.

From the previous section, we have that the underlying space of $[\mathcal{H}/\mathrm{PSL}(2, \mathbb{Z})]$ can be represented by the set F', inheriting the quotient topology. Since the set of singular points of F' is $\Sigma_{F'} = \{i, e^{2\pi i/3}\}$, [3, p. 14, 15] if we take the suborbifolds of F' defined by the subspaces

$$F_1 = \{z \in F' \mid Re(z) < -\frac{1}{4}\} \cup \{z \in F' \mid Re(z) > \frac{1}{4}\}$$

and

$$F_2 = \{z \in F' \mid -\frac{1}{3} < Re(z) < \frac{1}{3}\},$$

inheriting the subspace topology, by applying the van Kampen theorem version for orbifolds on F', F_1 and F_2, we obtain that

$$\pi_1^{\text{orb}}(F') = \pi_1^{\text{orb}}(F_1) *_{\pi_1^{\text{orb}}(F_1 \cap F_2)} \pi_1^{\text{orb}}(F_2). \quad (**)$$

The underlying space of $F_1 \cap F_2$ is defined by the subspace

$$\{z \in F' \mid -\frac{1}{3} < Re(z) < -\frac{1}{4}\} \cup \{z \in F' \mid \frac{1}{4} < Re(z) < \frac{1}{3}\}$$

inheriting the subspace topology. If $z = a + ib$ is an element of $\mathcal{H}$ such that $|z| = 1$ and $\gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then $\gamma z = -1/z = -\overline{z} = -a + ib$. Hence, we obtain that $\{z \in F' \mid -\frac{1}{3} < Re(z) < -\frac{1}{4},\ |z| = 1\}$ is (canonically) equivalent to $\{z \in F' \mid \frac{1}{4} < Re(z) < \frac{1}{3},\ |z| = 1\}$ via the map $z \mapsto -1/z$. Therefore, we have that $F_1 \cap F_2$ is a connected orbifold. Since every point in $F_1 \cap F_2$ has a trivial isotropic group, we have that $F_1 \cap F_2$ is a manifold. Because $F_1 \cap F_2$ is contractible, we get that $\pi_1^{\text{orb}}(F_1 \cap F_2)$ is the trivial group. Hence, $(**)$ becomes

$$\pi_1^{\text{orb}}(F') = \pi_1^{\text{orb}}(F_1) * \pi_1^{\text{orb}}(F_2). \quad (***)$$

The orbifold F_1 has only one element whose isotropy group is nontrivial, namely $w = e^{2\pi i/3}$. Its isotropy group Γ_w is the cyclic group generated by $\sigma = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$. Because $\sigma^3 = I$, we have that $\Gamma_w = C_3$. Since X_{F_1} is contractible and since F_1 has only one singular point, by Corollary 3.13 we have that $\pi_1^{\text{orb}}(F_1) \approx C_3$.

The isotropy group of $i \in F_2$ is a cyclic group of order 2. Since i is the only singular point of F_2, by a similar argument we have that $\pi_1^{\text{orb}}(F_2) \approx C_2$.

Therefore, $(***)$ gives us the desired result.

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