Normality of quotient spaces

For a quotient space, the separation axioms--even the Hausdorff property--are difficult to verify. We give here three situations in which the quotient space is not only Hausdorff, but normal.

Theorem G.1. Let $p : X \to Y$ be a closed quotient map. If X is normal, then Y is normal.

Proof. First we show that if A is a subset of Y, and N is an open set of X containing $p^{-1}(A)$, then there is an open set U of Y containing A such that $p^{-1}(U)$ is contained in N.

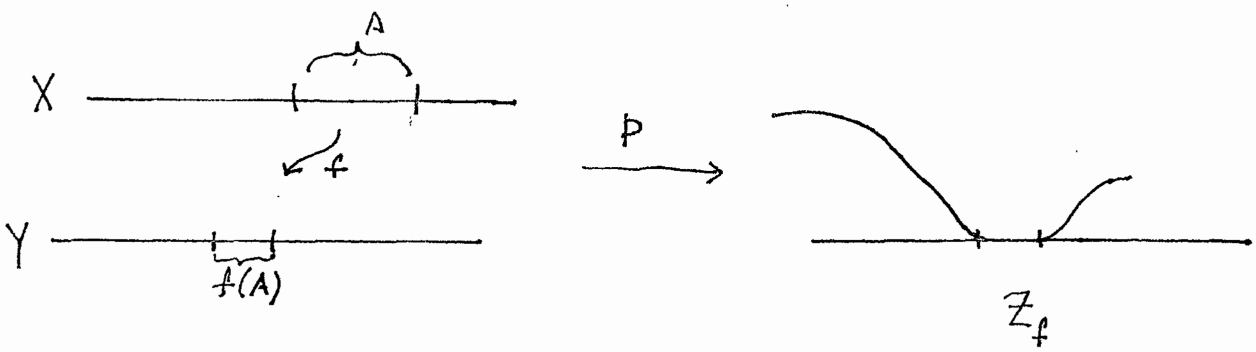
The proof is easy. The set $C = X - N$ is closed. The set $p(C)$ is closed and disjoint from A, so that the set $U = Y - p(C)$ is an open set of Y that contains A. If x is a point of U, then $p^{-1}(x)$ contains no point of C, so that it lies in N; thus $p^{-1}(U)$ is contained in N.

Now we verify normality of Y. Since one-point sets are closed in X and p is a closed map, one-point sets are closed in Y. Now let A and B be disjoint closed sets of Y. Since p is continuous, $p^{-1}(A)$ and $p^{-1}(B)$ are disjoint closed sets of X. Choose disjoint open sets N_1 and N_2 of X containing them. Let U_1 and U_2 be open sets of Y containing A and B, such that $p^{-1}(U_1)$ lies in N_1 and $p^{-1}(U_2)$ lies in N_2. Because N_1 and N_2 are disjoint, so are U_1 and U_2. □

Definition. Let X and Y be disjoint spaces; let A be a closed subset of X; and let $f : A \to Y$ be a continuous function. We define the adjunction space Z_f to be the quotient space obtained from the union of X and Y by identifying each point a of A with the point $f(a)$ and with all the points of $f^{-1}(f(a))$. Let $p : X \cup Y \to Z_f$ be the quotient map.

Now the map $p|Y$ is a continuous injection of Y into Z_f. We show that it is also a closed map. If C is a closed set of Y, then $p^{-1}(p(C))$ equals the union of C and $f^{-1}(C)$. The set C is closed in Y, so the set $f^{-1}(C)$ is closed in A and hence closed in X. Therefore, $p^{-1}(p(C))$ is closed in $X \cup Y$, so that $p(C)$ is closed in Z_f, by definition of the quotient topology.

It now follows that $p(Y)$ is a closed subspace of Z_f, and that $p|Y$ is a homeomorphism of Y with $p(Y)$.

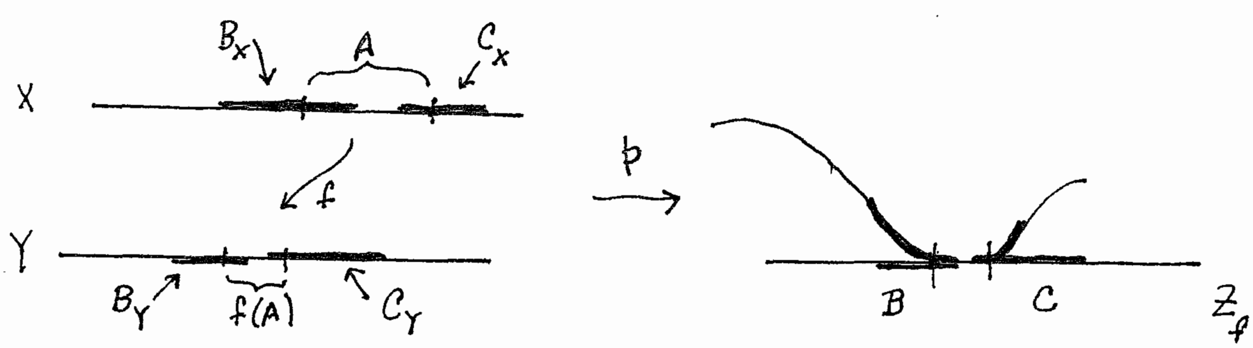


<u>Theorem G.2.</u> If X and Y are normal, then so is Z_f.

<u>Proof.</u> A direct proof, using the definition of normality, is a bit elaborate. (See [D], p. 145.) An easier proof uses the Tietze theorem, as we now show.

First, we note that Z_f is T_1. Let z be a point of Z_f. If z belongs to $p(Y)$, then $\{z\}$ is closed because one-point sets are closed in Y, and $p|Y : Y \to Z_f$ is a closed map. Otherwise, $p^{-1}(z)$ is a one-point set in X, and therefore closed; it follows from the definition of a quotient map that $\{z\}$ is closed.

Now let B and C be disjoint closed sets of Z_f. Let $B_X = p^{-1}(B) \cap X$ and $C_X = p^{-1}(C) \cap X$. Similarly, let $B_Y = p^{-1}(B) \cap Y$ and $C_Y = p^{-1}(C) \cap Y$.



Using normalilty of Y, choose a continuous function $g : Y \to [0,1]$ that equals 0 on B_Y and 1 on C_Y. Then define

$$h : A \cup B_X \cup C_X \to [0,1]$$

by setting $h = g \circ f$ on A, and $h = 0$ on B_X, and $h = 1$ on C_X. Because each of these three sets is closed in X and h is unambiguously defined when two of the sets overlap, h is continuous, by the pasting lemma. Using normality of X and the Tietze theorem, extend h to a continuous function $k : X \to [0,1]$. Then g and k together define a continuous function from $X \cup Y$ into $[0,1]$; it induces a continuous function

$$F : Z_f \to [0,1]$$

on the quotient space that equals 0 on B and 1 on C. The sets $F^{-1}([0,\frac{1}{2}))$ and $F^{-1}((\frac{1}{2},1])$ are then disjoint open sets about B and C, respectively. □

One application of adjunction spaces occurs in point-set topology, when one is studying absolute retracts. (See Exercise 8, p.224.) Another application occurs in algebraic topology, when one constructs a CW complex; we will discuss this application shortly.

<u>Definition</u>. Let X be a space and let $\{X_\alpha\}_{\alpha\in J}$ be a family of subspaces of X whose union is X. The topology of X is said to be <u>coherent</u> with the subspaces X_α if a set A is closed in X whenever $A \cap X_\alpha$ is closed in X_α for each α. (Or, equivalently, if a set U is open in X whenever $U \cap X_\alpha$ is open in X_α for each α.)

There is a strong connection between coherent topologies and quotient spaces. It is described as follows: To begin, let us give J the discrete topology and consider the product space $X \times J$. Then we consider the subspace of $X \times J$ that is the union of the subspaces $X_\alpha \times \{\alpha\}$, for all $\alpha \in J$. This space is called the <u>topological sum</u> (or sometimes the <u>disjoint union</u>) of the spaces X_α. It is denoted $\sum X_\alpha$. If we project $X \times J$ onto X, we obtain a continuous map

$$p : \sum X_\alpha \to X$$

which maps each space $X_\alpha \times \{\alpha\}$ by the obvious homeomorphism onto X_α. The map p is a quotient map if and only if the topology of X is coherent with the subspaces X_α. It follows that if X has the topology coherent with the subspaces X_α, then a map $f : X \to Y$ is continuous if and only if each of the functions $f|X_\alpha$ is continuous.

<u>Theorem G.3</u>. Let X be a space that is the union of countably many closed subspaces X_i, for $i \in Z_+$. Suppose the topology of X is coherent with these subspaces. If each X_i is normal, then so is X.

<u>Proof.</u> If p is a point of X, then $\{p\} \cap X_j$ is closed in X_i for each i, so $\{p\}$ is closed in X. Therefore X is a T_1 space.

Let A and B be closed disjoint sets in X. Define $Y_0 = A \cup B$, and for $n > 0$, define

$$Y_n = A \cup B \cup X_1 \cup \ldots \cup X_n .$$

Define a continuous function $f_0 : Y_0 \to [0,1]$ by letting it equal 0 on A and 1 on B. In general, suppose one is given a continuous function $f_n : Y_n \to [0,1]$. The space X_{n+1} is normal and $Y_n \cap X_{n+1}$ is closed in X_{n+1}. If g_n denotes the restriction of f_n to the subspace $Y_n \cap X_{n+1}$, we use the Tietze theorem to extend g_n to a continuous function $g : X_{n+1} \to [0,1]$. Because Y_n and X_{n+1} are closed subsets of Y_{n+1}, the functions f_n and g combine to define a continuous function

$$f_{n+1} : Y_{n+1} \to [0,1]$$

that is an extension of f_n. The functions f_n in turn combine to define a function $f : X \to [0,1]$ that equals 0 on A and 1 on B. <u>Because</u> X has the topology coherent with the subspaces X_n, the map f is continuous. □

<u>Example 1.</u> The preceding theorem does not extend to uncountable coherent unions. Given an element α of S_Ω, let X_α be the subspace consisting of all elements x of $\bar{S}$ such that $x \leq \alpha$. Then X_α is a closed interval in S_Ω, so it is compact.

The space $S_\Omega \times \bar{S}_\Omega$ is the union of the spaces $X_\alpha \times \bar{S}_\Omega$, each of which is compact Hausdorff and thus normal. We show that $S_\Omega \times \bar{S}_\Omega$, which is <u>not</u> normal, has the topology coherent with these subspaces.

Let U be a subset of $S_\Omega \times \bar{S}_\Omega$ such that $U \cap (X_\alpha \times \bar{S}_\Omega)$ is open in this subspace, for each α. Then the intersection

$$U \cap (S_\alpha \times \bar{S}_\Omega)$$

is open in $S_\alpha \times \bar{S}_\Omega$ for each α, and hence open in $S_\Omega \times \bar{S}_\Omega$. Since U is the union of the sets

$$U \cap (S_\alpha \times \bar{S}_\Omega),$$

it is open in $S_\Omega \times \bar{S}_\Omega$, as desired.

It is an interesting question to ask under what conditions coherent topologies exist. One has the following two theorems:

Theorem G.4. Let X be a set that is the union of the topological spaces X_α, for $\alpha \in J$. If there is a topological space X_T having X as its underlying set, such that each X_α is a subspace of X_T, then there is a topological space X_C such that each X_α is a subspace of X_C and the topology of X_C is coherent with the subspaces X_α. The topology of X_C is finer than that of X_T.

Proof. We define a set D to be closed in X_C if $D \cap X_\alpha$ is closed in X_α for each α. It is immediate that $\emptyset$ and X are closed. The required properties about unions and intersections follow from the equations

$$(D_1 \cup \dots \cup D_n) \cap X_\alpha = (D_1 \cap X_\alpha) \cup \dots \cup (D_n \cap X_\alpha) ,$$

$$(\bigcap_{D \in \mathcal{D}} D) \cap X_\alpha = \bigcap_{D \in \mathcal{D}} (D \cap X_\alpha),$$

where $\mathcal{D}$ is an arbitrary collection of closed sets.

Note that if E is closed in X_T, then $E \cap X_\alpha$ is closed in X_α for each α, so that E is closed in X_C. Thus the topology of X_C is finer than the topology of X_T.

What else is there to prove? We must prove that each X_α is a subspace of X_C. Isn't this obvious? Not quite. First, note that if A is closed in X_C, then $A \cap X_\alpha$ is closed in X_α by definition. Conversely, suppose B is closed in X_α. Because X_α is a subspace of X_T, we have $B = A \cap X_\alpha$ for some set A closed in X_T. Because the topology of X_C is finer than that of X_T, the set A is also closed in X_C. Thus $B = A \cap X_\alpha$ for some A closed in X_C, as desired. $\square$

Theorem G.5. Let X be a set that is the union of the topological spaces X_α, for $\alpha \in J$. If for each pair of indices α, β, the set $X_\alpha \cap X_\beta$ is closed in both X_α and X_β, and inherits the same topology from each of them, then X has a topology coherent with the subspaces X_α. Each X_α is a closed set in X in this topology.

Proof. Once again, we define a topological space X_C by declaring a set D to be closed in X_C if $D \cap X_\alpha$ is closed in X_α for each α. It is immediate that this is a topology on X.

We show that each space X_α is a closed subspace of X_C. First, if A is closed in X_C, then $A \cap X_\alpha$ is closed in X_α by definition of X_C. Conversely, let B be a closed set of X_α; we show B is closed in X_C. To do this, we must show that $B \cap X_\beta$ is closed in X_β for each β. Since B is closed in X_α, the set $B \cap X_\beta$ is closed in $X_\alpha \cap X_\beta$ because the latter is a subspace of X_α. Then $B \cap X_\beta$ is closed in X_β because $X_\alpha \cap X_\beta$ is a closed subspace of X_β. □

This theorem does not hold if the word "closed" is omitted from the hypothesis. There is an example of a set X that is the union of three spaces such that the intersection of any two of the spaces is a subspace of each of them; but there is no topology on X at all of which all three of the spaces are subspaces! (See [Mu], p. 213.)

Example 2. Consider $\mathbb{R}^\omega$ in the product topology. Let $\tilde{\mathbb{R}}^n$ denote the subspace of $\mathbb{R}^\omega$ consisting of all points $\underline{x} = (x_1, x_2, \ldots)$ such that $x_i = 0$ for $i > n$. Then one has the sequence of subspaces

$$\tilde{\mathbb{R}}^1 \subset \tilde{\mathbb{R}}^2 \subset \ldots ,$$

each of which is a closed subspace of the next. Their union is $\mathbb{R}^\infty$, which by Theorem G.5 has a topology coherent with the subspaces $\tilde{\mathbb{R}}^n$. Theorem G.3 implies that $\mathbb{R}^\infty$ is normal in this topology.

Now $\mathbb{R}^\infty$ also has several other topologies as well, ones that it inherits as a subspace of $\mathbb{R}^\omega$ in its various topologies. The subset $\tilde{\mathbb{R}}^n$ inherits its usual topology from each of these topologies on $\mathbb{R}^\infty$. Hence Theorem G.4 also applies to show that $\mathbb{R}^\infty$ has a topology coherent with the subspaces $\tilde{\mathbb{R}}^n$; this theorem also implies that the coherent topology is finer than each of these topologies on $\mathbb{R}^\infty$. Since the one derived from the box topology is the finest of these, one has the following:

Challenge question: Is the topology on $\mathbb{R}^\infty$ that is coherent with the subspaces $\tilde{\mathbb{R}}^n$ the same as the topology that $\mathbb{R}^\infty$ inherits as a subspace of $\mathbb{R}^\omega$ in the box topology?

Final remark. Here is a quick outline of how these notions are used in algebraic topology. (See §38 of [Mu] for more details.)

The unit ball in $\mathbb{R}^n$ is the subset of $\mathbb{R}^n$ consisting of all points whose euclidean distance from the origin is less than or equal to one; the unit sphere consists of those points for which this distance equals one. These spaces are denoted B^n and S^{n-1}, respectively. If there is a homeomorphism $h : B^n \to C$, then C is called an n-cell, and we denote by $\text{Bd}\, C$ the subspace $h(S^{n-1})$.

There is a class of spaces that is very important in algebraic topology called CW complexes; they were invented by J.H.C.Whitehead. In algebraic topology, one defines for a given space a number of groups, such as the homology groups $H_n(X)$, the cohomology groups $H^n(X)$, and the homotopy groups $\pi_n(X)$. Defining is one thing, but computing (or even getting useful information) is another. The structure of CW complex gives one a tool for dealing with these groups. CW complexes are quite versatile--many useful spaces, such as the Grassman manifolds we shall mention shortly, have the structure of a CW complex. On the other hand, if one has some prescribed groups and wants to find a space for which the homology groups, say, are isomorphic to these groups, one can construct a CW complex that will do the job.

A CW complex is constructed as follows: One begins with a discrete space, which we call X^0. Then one takes a collection of disjoint 1-cells C_α, and a family of continuous maps $f_\alpha : \text{Bd}\, C_\alpha \to X^0$. One forms the topological sum $\sum C_\alpha$, and uses the maps f_α to define a continuous map $f : \sum \text{Bd}\, C_\alpha \to X^0$. One forms the adjunction space obtained from $X^0 \cup \sum C_\alpha$ by means of this map. This space is denoted X^1; it is called a 1-dimensional CW complex.

So far, so good. Now one takes a collection of disjoint 2-cells D_β, and a continuous map $g : \sum \text{Bd}\, D_\beta \to X^1$, and forms an adjunction space from $\sum D_\beta$ and X^1 by means of this map. This space is denoted X^2 and is a 2-dimensional CW complex.

It is clear how to continue. One has eventually an n-dimensional CW complex X^n, for each n. Are we finished? No. Recall that in the construction of the adjunction space X^n, the projection map defines a homeomorphism of X^{n-1} with a closed subspace of X^n. We normally identify X^{n-1} with this closed subspace of X^n.

With this convention, we now have a sequence of spaces

$$X^0 \subset X^1 \subset \ \dots \subset X^n \subset \dots$$

each of which is a closed subspace of the next. Their union is given the topology coherent with these subspaces; it is called an (infinite-dimensional) CW complex.

In order to work with the space we obtain, it is essential that it be a Hausdorff space. (Basically, so we know that compact sets are closed.) The theorems we have proved in this section do much more than that; they show that every CW complex is normal.

Final final remark. We have talked a lot about how quotient spaces are used in algebraic topology. Let us close by giving an example of how they are used in differential geometry.

A very important space in differential geometry is the space $G_{n,k}$ of k-dimensional vector subspaces of $\mathbb{R}^n$. It is called the Grassman manifold of k-planes in n-space. The space $G_{n,1}$ is thus the space of all lines through the origin in $\mathbb{R}^n$. It is fairly intuitive what one means by saying that two k-planes are "close" to one another, but how does one topologize this space rigorously? One topologizes it as a quotient space.

To be specific, let $V_{n,k}$ be the set of all k by n matrices, where $k \leq n$, whose rows are orthonormal vectors. Such a matrix satisfies the equation $AA^t = I_k$. There is an obvious topology on this set, for it can be considered as a subspace of $\mathbb{R}^{kn}$. In this topology, it is compact, for it is closed and bounded, as the equation $AA^t = I_k$ shows. Let $p : V_{n,k} \to G_{n,k}$ be the map that sends each matrix to the k-dimensional vector subspace of $\mathbb{R}^n$ that is spanned by its rows. We topologize $G_{n,k}$ by requiring p to be a quotient map.

Because p is continuous, it is immediate that $G_{n,k}$ is compact. The next question is this: Is it Hausdorff? The answer is "yes," because the map p is in fact a closed map, so Theorem G.1 applies.

To show p is closed, we examine first what the relationship is between two matrices A and B whose rows span the same vector subspace of $\mathbb{R}^n$. This occurs precisely when each row of A equals a linear combination of the rows of B, and conversely. This statement can be expressed by the matrix equation $A = CB$, where C is a nonsingular k by k matrix. It follows that C satisfies the equation $CC^t = I_k$, which means that C belongs to $V_{k,k}$. A quick computation with matrices verifies this fact: The equation $A = CB$, along with the equations $AA^t = BB^t = I_k$, implies that

$$AB^t = C \quad \text{and} \quad I_k = C(BA^t).$$

The first equation gives us, by transposing, the equation $BA^t = C^t$; substituting this result into the second equation gives us the equation

$$I_k = CC^t.$$

Now we show that p is a closed map. If S is a closed set in $V_{n,k}$, then the set $p^{-1}(p(S))$ is the set of all matrices of the form CA, where C belongs to $V_{k,k}$ and A is an element of S. Thus $p^{-1}(p(S))$ is the image of $V_{k,k} \times S$ under the map given by matrix multiplication. Now $V_{k,k}$ is compact and S is compact (being closed in $V_{n,k}$). Their cartesian product is compact, so the image under matrix multiplication (which is continuous) is also compact and therefore a closed subset of $V_{n,k}$. By definition of the quotient topology, it follows that $p(S)$ is closed in $G_{n,k}$, as desired.

There is of course a great deal more to say about Grassman manifolds. The space $G_{n,k}$ is in fact a manifold (as the terminology implies); it is a manifold of dimension $k(n - k)$.

If we replace $\mathbb{R}^n$ throughout by $\tilde{\mathbb{R}}^n$, then there is the obvious inclusion of $\tilde{\mathbb{R}}^n$ into $\tilde{\mathbb{R}}^{n+1}$; it gives rise to an inclusion map of $V_{n,k}$ into $V_{n+1,k}$. This in turn induces a continuous injective map on the quotient spaces

$$i: G_{n,k} \to G_{n+1,k} .$$

Since all the spaces involved are compact Hausdorff, we can thus consider $G_{n,k}$ to be a closed subspace of $G_{n+1,k}$. If now we take the union of the spaces

$$G_{k,k} \subset G_{k+1,k} \subset \ldots \subset G_{n,k} \subset \ldots ,$$

one has the space of all k planes in $\mathbb{R}^\infty$. As you would expect, we give it the coherent topology. And Theorem G.3 implies that this space is normal!