LECTURE 16

Vanishing of Tate Cohomology Groups

Recall that we reduced (cohomological) local class field theory to the following statement: for a finite Galois extension L/K of nonarchimedean local fields with Galois group G, we have

$$\left((L \otimes_K K^{\mathrm{unr}})^\times\right)^{\mathrm{t}G} \simeq 0,$$

i.e., this complex is acyclic. To show this vanishing, we will prove a general theorem (due to Tate) about the vanishing of Tate cohomology, which makes the above more tractable. Thus, we ask: given a complex X of G-modules, what conditions guarantee that $X^{\mathrm{t}G}$ is acyclic? The prototypical such result is the following:

THEOREM 16.1. *For a cyclic group G, $X^{\mathrm{t}G}$ is acyclic if and only if*

$$\hat{H}^0(G, X) = 0 = \hat{H}^1(G, X).$$

PROOF. $X^{\mathrm{t}G}$ can be computed be a 2-periodic complex. Note that any two values of distinct parity (such as consecutive values) would suffice. □

Our main results in this lecture are the following:

THEOREM 16.2. *Theorem 16.1 holds also if G is a p-group (i.e., $\#G = p^n$, for some prime p and $n \geq 0$).*

From here, we will deduce the next result:

THEOREM 16.3. *Suppose that for every prime p and every p-Sylow subgroup $G_p \subseteq G$, $\hat{H}^0(G_p, X) = 0 = \hat{H}^1(G_p, X)$. Then $X^{\mathrm{t}G}$ is acyclic.*

REMARK 16.4. In general, it's not true that vanishing in two consecutive degrees is sufficient for any finite group G. Also, in practice, one often verifies the vanishing of Tate cohomology in two consecutive subgroups for every subgroup of G, and not just p-Sylow ones.

In the following lectures, we will deduce local class field theory from here.

We begin by proving Theorem 16.2. Throughout the following, let G be a p-group.

PROPOSITION 16.5. *Let X be a complex of $\mathbb{F}_p[G]$-modules. If $\hat{H}^0(G, X) = 0$, then $X^{\mathrm{t}G}$ is acyclic.*

Note that we only need vanishing in one degree here! For this, we first recall the following fact.

LEMMA 16.6. *Let V be a non-zero $\mathbb{F}_p[G]$-module. Then $V^G \neq 0$.*

PROOF. Without loss of generality, we may assume that V is finite-dimensional over $\mathbb{F}_p$, since V is clearly the union of its finite-dimensional G-submodules. Then

$\#V = p^r$ for some $0 < r < \infty$. Every G-orbit of V either has size 1 or divisible by p (since they must divide $\#G$ as the orbit is isomorphic to the quotient of G by the stabilizer). Since $\{0\}$ is a G-orbit of size 1, there must be another (since the sizes of all the G-orbits must sum to p^r), that is, some $v \in V^G \setminus \{0\}$. □

PROOF (OF PROPOSITION 16.5). **Step 1.** First, we claim that $\hat{H}^0(G, X \otimes V) = 0$ for every finite-dimensional G-representation V over $\mathbb{F}_p$. Here G acts via the "diagonal action," i.e., on $(X \otimes V)^i = X^i \otimes V$ via $g \cdot (x \otimes v) := gx \otimes gv$. We proceed by induction on $\dim_{\mathbb{F}_p} V$. By the previous lemma, we have a short exact sequence of $\mathbb{F}_p[G]$-modules

$$0 \to V^G \to V \to W \to 0$$

with $\dim W < \dim V$. This gives

$$\mathrm{hCoker}(V^G \otimes X \to V \otimes X) \simeq W \otimes X.$$

Since $\hat{H}^0(G, W \otimes X) = 0$ by the inductive hypothesis, and

$$\hat{H}^0(G, V^G \otimes X) = \hat{H}^0\big(G, \bigoplus_{\dim V^G} X\big) = \bigoplus_{\dim V^G} \hat{H}^0(G, X) = 0$$

by assumption on X, the long exact sequence on Tate cohomology gives $\hat{H}^0(G, V \otimes X)$, as desired.

Step 2. We now show vanishing in negative degrees. Consider the short exact sequence

$$0 \to V_1 \to \mathbb{F}_p[G] \xrightarrow{\epsilon} \mathbb{F}_p \to 0,$$

where V_1 is defined as the kernel of ϵ, analogously to what we called I_G with $\mathbb{F}_p$ replaced by $\mathbb{Z}$. Let $\underline{X}$ be X with the trivial G-action. Then

$$\begin{gathered} \underline{X} \otimes \mathbb{F}_p[G] \to X \otimes \mathbb{F}_p[G] \\ x \otimes g \mapsto gx \otimes g \end{gathered}$$

is a G-equivariant map, and a bijection, hence an isomorphism. Indeed,

$$h(x \otimes g) = x \otimes hg \mapsto hgx \otimes hg = h(gx \otimes g).$$

In Problem 1(e) of Problem Set 7, it was proven that $(\underline{X} \otimes \mathbb{F}_p[G])^{\mathrm{t}G}$ is acyclic, hence $(X \otimes \mathbb{F}_p[G])^{\mathrm{t}G}$ is as well. Thus, the long exact sequence on Tate cohomology gives

$$\hat{H}^{i-1}(G, X) \simeq \hat{H}^i(G, X \otimes V_1).$$

We've seen in Step 1 that the right-hand side vanishes for $i = 0$, therefore $\hat{H}^{-1}(G, X) = 0$. Iterating, we get $\hat{H}^i(G, X) = 0$ for all $i \leq 0$.

Step 3. To show vanishing in positive degrees, note that we have an exact sequence

$$0 \to \mathbb{F}_p \xrightarrow{1 \mapsto \sum_{g \in G} g} \mathbb{F}_p[G] \to V_2 \to 0,$$

where V_2 is defined to be the cokernel as before. The same logic gives

$$\hat{H}^i(G, X \otimes V_2) \simeq \hat{H}^{i+1}(G, X),$$

and so Step 1 again shows that $\hat{H}^i(G, X) = 0$ for all $i \geq 0$. □

Proof (of Theorem 16.2). Define $X/p := \mathrm{hCoker}(X \xrightarrow{\times p} X)$; note that this is not the same as modding out all terms by p. Note that, as a complex of $\mathbb{Z}[G]$-modules, X/p is quasi-isomorphic to a complex of $\mathbb{F}_p[G]$-modules. Since X is only defined up to quasi-isomorphism, we may assume it is projective (in particular, flat) as a complex of $\mathbb{Z}[G]$-modules. Thus, we have a quasi-isomorphism

$$X/p = X \otimes_{\mathbb{Z}[G]} \mathrm{hCoker}(\mathbb{Z}[G] \xrightarrow{\times p} \mathbb{Z}[G]) \simeq X \otimes_{\mathbb{Z}[G]} \mathbb{F}_p[G],$$

where the right-hand side computes X modded out by p term-wise. We then have a long exact sequence

$$\to \hat{H}^i(G,X) \xrightarrow{\times p} \hat{H}^i(G,X) \to \hat{H}^i(G,X/p) \to \hat{H}^{i+1}(G,X) \xrightarrow{\times p} \hat{H}^{i+1}(G,X) \to$$

and so setting $i=0$, we obtain $\hat{H}^0(G,X/p) = 0$ by assumption. Thus, $(X/p)^{\mathrm{t}G}$ is acyclic by Proposition 16.5. It follows that $\hat{H}^i(G,X/p) = 0$ for all i, and therefore, multiplication by p is an isomorphism on $\hat{H}^i(G,X)$ for each i. But as shown in Problem 2(c) of Problem Set 7, multiplication by $\#G$ is zero on $\hat{H}^i(G,X)$. Since G is a p-group, this is only possible if $\hat{H}^i(G,X) = 0$ for all i. □

Proof (of Theorem 16.3). Since as mentioned above, multiplication by $\#G$ is the zero map on $\hat{H}^i(G,X)$, it follows that $\hat{H}^i(G,X)$ is $\#G$-torsion. Thus, it suffices to show that $\hat{H}^i(G,X)[p] = 0$ for all p (i.e., the p-torsion of $\hat{H}^i(G,X)$ vanishes).

Recall that for every subgroup H of G, there are restriction and inflation maps $X^{\mathrm{t}G} \to X^{\mathrm{t}H}$ and $X^{\mathrm{t}H} \to X^{\mathrm{t}G}$ respectively, whose composition as an endomorphism of $X^{\mathrm{t}G}$ is homotopic to multiplication by the index $[G:H]$.

Applying this to a p-Sylow subgroup $H = G_p$ of G and taking cohomology, we obtain maps

$$\hat{H}^i(G,X)[p] \subset \hat{H}^i(G,X) \to \hat{H}^i(G_p,X) \to \hat{H}^i(G,X)$$

whose composition is multiplication by $[G:G_p]$, which is prime to p by definition. Thus, it is an isomorphism when restricted to $\hat{H}^i(G,X)[p]$, and in particular, $\hat{H}^i(G,X)[p] \to \hat{H}^i(G_p,X)$ is injective. But by Theorem 16.2, $\hat{H}^i(G_p,X) = 0$ for all i, which yields the desired result. □

18.786 Number Theory II: Class Field Theory
Spring 2016