Preface

In the beginning group theory dealt with finite groups, for example permutation groups of the type which appear in Galois theory of field extensions.

However, the step from permutation groups to transformation groups is a simple and natural one. Consider for example the set of isometrics of the plane $\mathbf{R}^2$. Clearly, they form a group under composition; we denote this group $\mathcal{M}(2)$ (M suggesting motion). We can now *parameterize* this group. If $g \in \mathcal{M}(2)$ let t be a translation such that $g \cdot 0 = t \cdot 0$ ($g \cdot x$ denotes the image of x under g). Then $t^{-1}g \cdot 0 = 0$. Let e_1, e_2 denote the coordinate vectors $(1,0)$ and $(0,1)$. Choose a rotation k around the origin 0 such that $k \cdot e_1 = t^{-1}g \cdot e_1$. Then $h = k^{-1}t^{-1}g$ fixes both 0 and e_1. Since $h \cdot e_2$ lies on circles with centers 0 and e_1 we have either $he_2 = e_2$ or $rh \cdot e_2 = e_2$ where r is the reflection in the x-axis. Let $p \in \mathbf{R}^2$. In the first case $h \cdot p$ lies on circles with centers in 0, e_1 and e_2 so $h \cdot p = p$. In the second case $rh \cdot p = p$ for the same reason. We conclude that $g = ts$ where s is in the orthogonal group $\mathbf{O}(2)$ which is generated by the rotations $z \to e^{i\theta}z$ and the reflection $z \to \bar{z}$. Thus $\mathcal{M}(2)$ is described by three parameters.

Let $\mathbf{O}(n)$ denote the orthogonal group in $\mathbf{R}^n$ that is $g \in \mathbf{O}(n)$ if and only if $\langle g \cdot u, g \cdot v \rangle = \langle u, v \rangle$ for all vectors u and v, $\langle \, , \, \rangle$ denoting the scalar product. Consider the case $n = 3$. Each $k \in \mathbf{O}(3)$ has a real eigenvalue λ and $|\lambda| = 1$. Thus k maps a line ℓ into itself and leaves $\ell^\perp$ invariant. Also $k|\ell^\perp \in \mathbf{O}(2)$ so we get a parameterization of $\mathbf{O}(3)$ by combining parameterization of ℓ (coordinates in S^2) and the angle of rotation in $\ell^\perp$. Thus one can develop calculus (integration and differentiation) on such groups.

More generally if M is a Riemannian manifold the group $I(M)$ of its isometries can be parameterized (in terms of the geodesics in M) in such a way that the group operations are expressed analytically in terms of these parameters.

This brings us to the definition of a Lie group as a manifold G such that the group operation $(g,h) \to g^{-1}h$ is analytic from $G \times G$ to G.

Lie had the idea back in 1874 to try to create a theory for transformation groups which might do for differential equations what Galois theory did for algebraic equations. Here is one of Lie's theorems (1874).

Let φ_t be a group of diffeomorphisms of $\mathbf{R}^2$ depending on 1-parameter t, so $\varphi_0 = I$, $\varphi_{s+t} = \varphi_s \cdot \varphi_t$. It is said to leave the equation

$$\frac{dy}{dx} = \frac{Y(x,y)}{X(x,y)}$$

stable if each φ_t permutes the solutions. Consider the vector field

$$\Phi_p = \left\{ \frac{d(\varphi_t \cdot p)}{dt} \right\}_{t=0} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} .$$

Theorem 1. *If φ_t leaves the equation stable then the function*

$$(X\eta - Y\xi)^{-1}$$

is an integrating factor for $X\,dy - Y\,dx$ i.e., $\frac{Y\,dy - Y\,dx}{Y\eta - Y\xi} = dU$ and thus $U(x,y) = C$ is the solution.

Example (i)

$$\frac{dy}{dx} = \frac{y + x(x^2+y^2)}{x - y(x^2+y^2)} .$$

This can be written

$$\frac{\frac{dy}{dx} - \frac{y}{x}}{1 + \frac{y}{x}\frac{dy}{dx}} = x^2 + y^2 . \tag{1}$$

Recall that dy/dx and y/x represent the slopes of the tangent and the radius vector, respectively, and also the formula

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} .$$

The equation (1) thus shows that the angle between the two is constant on each circle through 0. Thus each rotation

$$\varphi_t : (x,y) \to (x\cos t - y\sin t\,,\ x\sin t + y\cos t)$$

leaves the equation stable. Now

$$\left\{ \frac{d\varphi_t(x,y)}{dt} \right\}_{t=0} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

so according to the theorem,

$$\frac{[x - y(x^2+y^2)]\,dy - [y + x(x^2+y^2)]\,dx}{(x - y(x^2+y^2))x + [y = x(x^2+y^2)]y}$$

is a differential dU and we see easily that we can take

$$U = \arctan\left(\frac{y}{x}\right) - \frac{x^2+y^2}{2} ,$$

so solution is $y = x\tan\left(\frac{x^2+y^2}{2} + C\right)$.

(ii) Another example (Lie 1874).

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}\,.$$

Transformation in (x, t, u) leaves equation stable if it permutes the solutions. Lie determined this group. It is a product of a 6-dim group and an infinite-dimensional group. Most subgroups of these are rather trivial, but one is important. It shows that if $u = f(x,t)$ is a solution then so is

$$U^{(s)}(x,t) = \frac{1}{\sqrt{1+4st}} e^{\frac{-sx^2}{1+4st}} f\left(\frac{x}{1+4st}, \frac{t}{1+4st}\right).$$

(iii) Another example: $\frac{du}{dx} = \left(\frac{d^2v}{dx^2}\right)^2$ has stability group the exceptional group G_2 .

Theorem 1 led Lie to an intensive study of transformation groups. Generalizing the φ_t above he considered transformations $T(x_1, \dots, x_n) \to (x'_1, \dots, x'_n)$ where

$$x'_i = f_i(x_1, \dots x_n\,;\, t_1, \dots, t_r)$$

depending effectively on r parameters t_i, the f_i being smooth functions and the Jacobian $(\partial f_i / \partial t_k)$ of rank r. Also the identity map I corresponds to $(t_1, \dots, t_r) = (0, \dots, 0)$ and if S corresponds to $(s_1, \dots, s_r)$ then TS^{-1} is for small t_i, s_j given by

$$x'_i = f_i(x_1, \dots, x_n\,;\, u_1, \dots, u_r)$$

where the u_k are analytic functions of t_i, s_j. Generalizing the Φ above Lie considered the vector fields

$$T_k = \sum_{i=1}^{n} \left(\frac{\partial f_i}{\partial t_k}\right)_{t=0} \frac{\partial}{\partial x_i}\,,\quad 1 \leqslant k \leqslant r$$

and proved the following fundamental result.

Theorem. *The bracket*

$$[T_k, T_\ell] = T_k \circ T_\ell - T_\ell \circ T_k$$

satisfies

$$[T_k, T_\ell] = \sum_1^r c^p_{k\ell} T_p$$

where the $c^p_{k\ell}$ *are constants satisfying*

$$c^p_{k\ell} = -c^p_{\ell k}\,, \quad \sum_q (c^p_{kq} c^q_{\ell m} + c^p_{mq} c^q_{k\ell} + c^p_{\ell q} c^q_{mk}) = 0\,.$$

This leads to the concept of a Lie algebra.

Definition. A Lie algebra is a vector space V over a field such that there is a bilinear map $(X, Y) \to [X, Y]$ of $V \times V$ into V satisfying

$$[X, Y] = -[Y, X]$$
$$[X, [Y, Z]] + [Y, [Z, X]] + [X, [Y, Z]] = 0\,.$$

If V has a finite dimension with basis (X_i) and we define c_{ij}^k by

$$[X_i, X_j] = \sum_k c_{ij}^k X_k\,. \tag{*}$$

then the above relations are satisfied. Conversely, if (X_i) is a basis of a vector space V and we define a rule of composition by (*) then the relations imply that V is a Lie algebra.

Lie's proof was complicated; we can now give a simpler proof. Lie also proved a converse (Lie's 2^{nd} and 3^{rd} theorems), at least locally.

Historically the theory of Lie groups started with applications to differential equations in mind. This remained a project of Lie. Primarily through the work of Killing and Cartan, Lie group theory was developed on its own but got tied up with differential geometry, particularly with the advent of fibre bundles.

In the mid-twenties applications to physics came about and this has continued to our days. A journal on mathematical physics will have Lie groups on most pages. A very important application came in the mid-fifties to the theory of finite simple groups. Lie theory played a major role in the classification of these finite groups. There has also been a kind of renaissance in applications to differential equations; also in number theory and automorphic forms. So Lie groups do indeed play a central role in many fields of mathematics.