4. The Macdonald-Mehta integral

4.1. **Finite Coxeter groups and the Macdonald-Mehta integral.** Let W be a finite Coxeter group of rank r with real reflection representation $\mathfrak{h}_{\mathbb{R}}$ equipped with a Euclidean W-invariant inner product $(\cdot,\cdot)$. Denote by $\mathfrak{h}$ the complexification of $\mathfrak{h}_{\mathbb{R}}$. The reflection hyperplanes subdivide $\mathfrak{h}_{\mathbb{R}}$ into $|W|$ chambers; let us pick one of them to be the dominant chamber and call its interior D. For each reflection hyperplane, pick the perpendicular vector $\alpha\in\mathfrak{h}_{\mathbb{R}}$ with $(\alpha,\alpha)=2$ which has positive inner products with elements of D, and call it the positive root corresponding to this hyperplane. The walls of D are then defined by the equations $(\alpha_i,v)=0$, where α_i are simple roots. Denote by $\mathcal{S}$ the set of reflections in W, and for a reflection $s\in\mathcal{S}$ denote by α_s the corresponding positive root. Let

$$\delta(\mathbf{x})=\prod_{s\in\mathcal{S}}(\alpha_s,\mathbf{x})$$

be the corresponding discriminant polynomial. Let $d_i, i=1,\ldots,r$, be the degrees of the generators of the algebra $\mathbb{C}[\mathfrak{h}]^W$. Note that $|W|=\prod_i d_i$.

Let $H_{1,c}(W,\mathfrak{h})$ be the rational Cherednik algebra of W. Here we choose $c=-k$ as a constant function. Let $M_c=M_c(\mathbb{C})$ be the polynomial representation of $H_{1,c}(W,\mathfrak{h})$, and β_c be the contravariant form on M_c defined in Section 3.12. We normalize it by the condition $\beta_c(1,1)=1$.

Theorem 4.1. (i) *(The Macdonald-Mehta integral) For* $\mathrm{Re}\,(k)\geq 0$*, one has*

$$(2\pi)^{-r/2}\int_{\mathfrak{h}_{\mathbb{R}}}\mathbf{e}^{-(\mathbf{x},\mathbf{x})/2}|\delta(\mathbf{x})|^{2k}\mathrm{d}\mathbf{x}=\prod_{i=1}^{r}\frac{\Gamma(1+kd_i)}{\Gamma(1+k)}. \tag{4.1}$$

(ii) *Let* $b(k):=\beta_c(\delta,\delta)$*. Then*

$$b(k)=|W|\prod_{i=1}^{r}\prod_{m=1}^{d_i-1}(kd_i+m).$$

For Weyl groups, this theorem was proved by E. Opdam [Op1]. The non-crystallographic cases were done by Opdam in [Op2] using a direct computation in the rank 2 case (reducing (4.1) to the beta integral by passing to polar coordinates), and a computer calculation by F. Garvan for H_3 and H_4.

Example 4.2. In the case $W=\mathfrak{S}_n$, we have the following integral (the Mehta integral):

$$(2\pi)^{-(n-1)/2}\int_{\{\mathbf{x}\in\mathbb{R}^n|\sum_i x_i=0\}}\mathbf{e}^{-(\mathbf{x},\mathbf{x})/2}\prod_{i\neq j}|x_i-x_j|^{2k}\mathrm{d}\mathbf{x}=\prod_{d=2}^{n}\frac{\Gamma(1+kd)}{\Gamma(1+k)}.$$

In the next subsection, we give a uniform proof of Theorem 4.1 which is given in [E2]. We emphasize that many parts of this proof are borrowed from Opdam's previous proof of this theorem.

4.2. **Proof of Theorem 4.1.**

Proposition 4.3. *The function b is a polynomial of degree at most $|\mathcal{S}|$, and the roots of b are negative rational numbers.*

Proof. Since δ has degree $|\mathcal{S}|$, it follows from the definition of b that it is a polynomial of degree $\leq |\mathcal{S}|$.

Suppose that $b(k) = 0$ for some $k \in \mathbb{C}$. Then $\beta_c(\delta, P) = 0$ for any polynomial P. Indeed, if there exists a P such that $\beta_c(\delta, P) \neq 0$, then there exists such a P which is antisymmetric of degree $|\mathcal{S}|$. Then P must be a multiple of δ which contradicts the equality $\beta_c(\delta, \delta) = 0$.

Thus, M_c is reducible and hence has a singular vector, i.e. a nonzero homogeneous polynomial f of positive degree d living in an irreducible representation τ of W killed by y_a. Applying the element $\mathbf{h} = \sum_i x_{a_i} y_{a_i} + r/2 + k \sum_{s \in \mathcal{S}} s$ to f, we get

$$k = -\frac{d}{m_\tau},$$

where m_τ is the eigenvalue of the operator $T := \sum_{s \in \mathcal{S}} (1 - s)$ on τ. But it is clear (by computing the trace of T) that $m_\tau \geq 0$ and $m_\tau \in \mathbb{Q}$. This implies that any root of b is negative rational. $\square$

Denote the Macdonald-Mehta integral by $F(k)$.

Proposition 4.4. *One has*

$$F(k+1) = b(k) F(k).$$

Proof. Let $\mathbf{F} = \sum_i y_{a_i}^2 / 2$. Introduce *the Gaussian inner product* on M_c as follows:

Definition 4.5. *The Gaussian inner product γ_c on M_c is given by the formula*

$$\gamma_c(v, v') = \beta_c(\exp(\mathbf{F}) v, \exp(\mathbf{F}) v').$$

This makes sense because the operator $\mathbf{F}$ is locally nilpotent on M_c. Note that δ is a nonzero W-antisymmetric polynomial of the smallest possible degree, so $(\sum y_{a_i}^2)\delta = 0$ and hence

$$\gamma_c(\delta, \delta) = \beta_c(\delta, \delta) = b(k). \tag{4.2}$$

For $a \in \mathfrak{h}$, let $x_a \in \mathfrak{h}^* \subset H_{1,c}(W, \mathfrak{h})$, $y_a \in \mathfrak{h} \subset H_{1,c}(W, \mathfrak{h})$ be the corresponding generators of the rational Cherednik algebra.

Proposition 4.6. *Up to scaling, γ_c is the unique W-invariant symmetric bilinear form on M_c satisfying the condition*

$$\gamma_c((x_a - y_a)v, v') = \gamma_c(v, y_a v'), \ a \in \mathfrak{h}.$$

Proof. We have

$$\begin{aligned} & \gamma_c((x_a - y_a)v, v') = \beta_c(\exp(\mathbf{F})(x_a - y_a)v, \exp(\mathbf{F})v') = \beta_c(x_a \exp(\mathbf{F})v, \exp(\mathbf{F})v') \\ = \ & \beta_c(\exp(\mathbf{F})v, y_a \exp(\mathbf{F})v') = \beta_c(\exp(\mathbf{F})v, \exp(\mathbf{F}) y_a v') = \gamma_c(v, y_a v'). \end{aligned}$$

Let us now show uniqueness. If γ is any W-invariant symmetric bilinear form satisfying the condition of the Proposition, then let $\beta(v, v') = \gamma(\exp(-\mathbf{F})v, \exp(-\mathbf{F})v')$. Then β is contravariant, so it's a multiple of β_c, hence γ is a multiple of γ_c. $\square$

Now we will need the following known result (see [Du2], Theorem 3.10).

Proposition 4.7. *For* $\mathrm{Re}\,(k) \geq 0$ *we have*

$$\gamma_c(f, g) = F(k)^{-1} \int_{\mathfrak{h}_{\mathbb{R}}} f(\mathbf{x}) g(\mathbf{x}) \mathrm{d}\mu_c(\mathbf{x}) \tag{4.3}$$

where

$$\mathrm{d}\mu_c(\mathbf{x}) := \mathbf{e}^{-(\mathbf{x},\mathbf{x})/2} |\delta(\mathbf{x})|^{2k} \mathrm{d}\mathbf{x}.$$

Proof. It follows from Proposition 4.6 that γ_c is uniquely, up to scaling, determined by the condition that it is W-invariant, and $y_a^\dagger = x_a - y_a$. These properties are easy to check for the right hand side of (4.3), using the fact that the action of y_a is given by Dunkl operators. □

Now we can complete the proof of Proposition 4.4. By Proposition 4.7, we have

$$F(k + 1) = F(k)\gamma_c(\delta, \delta),$$

so by (4.2) we have

$$F(k + 1) = F(k)b(k).$$

□

Let

$$b(k) = b_0 \prod (k + k_i)^{n_i}.$$

We know that $k_i > 0$, and also $b_0 > 0$ (because the inner product β_0 on real polynomials is positive definite).

Corollary 4.8. *We have*

$$F(k) = b_0^k \prod_i \left(\frac{\Gamma(k + k_i)}{\Gamma(k_i)} \right)^{n_i}.$$

Proof. Denote the right hand side by $F_*(k)$ and let $\phi(k) = F(k)/F_*(k)$. Clearly, $\phi(0) = 1$. Proposition 4.4 implies that $\phi(k)$ is a 1-periodic positive function on $[0, \infty)$. Also by the Cauchy-Schwarz inequality,

$$F(k)F(k') \geq F((k + k')/2)^2,$$

so $\log F(k)$ is convex for $k \geq 0$. This implies that $\phi = 1$, since $(\log F_*(k))'' \to 0$ as $k \to +\infty$. □

Remark 4.9. The proof of this corollary is motivated by the standard proof of the following well known characterization of the Γ function.

Proposition 4.10. *The* Γ *function is determined by three properties:*

(i) $\Gamma(x)$ *is positive on* $[1, +\infty)$ *and* $\Gamma(1) = 1$*;*
(ii) $\Gamma(x + 1) = x\Gamma(x)$*;*
(iii) $\log \Gamma(x)$ *is a convex function on* $[1, +\infty)$.

Proof. It is easy to see from the definition $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ that the Γ function has properties (i) and (ii); property (iii) follows from this definition and the Cauchy-Schwarz inequality.

Conversely, suppose we have a function $F(x)$ satisfying the above properties, then we have $F(x) = \phi(x)\Gamma(x)$ for some 1-periodic function $\phi(x)$ with $\phi(x) > 0$. Thus, we have

$$(\log F)'' = (\log \phi)'' + (\log \Gamma)''.$$

Since $\lim_{x\to+\infty}(\log\Gamma)''=0$, $(\log F)''\geq 0$, and ϕ is periodic, we have $(\log\phi)''\geq 0$. Since $\int_n^{n+1}(\log\phi)''\mathrm{d}x=0$, we see that $(\log\phi)''\equiv 0$. So we have $\phi(x)\equiv 1$. $\square$

In particular, we see from Corollary 4.8 and the multiplication formulas for the Γ function that part (ii) of Theorem 4.1 implies part (i).

It remains to establish (ii).

Proposition 4.11. *The polynomial b has degree exactly $|\mathcal{S}|$.*

Proof. By Proposition 4.3, b is a polynomial of degree at most $|\mathcal{S}|$. To see that the degree is precisely $|\mathcal{S}|$, let us make the change of variable $\mathbf{x}=k^{1/2}\mathbf{y}$ in the Macdonald-Mehta integral and use the steepest descent method. We find that the leading term of the asymptotics of $\log F(k)$ as $k\to+\infty$ is $|\mathcal{S}|k\log k$. This together with the Stirling formula and Corollary 4.8 implies the statement. $\square$

Proposition 4.12. *The function*

$$G(k):=F(k)\prod_{j=1}^{r}\frac{1-\mathbf{e}^{2\pi\mathbf{i}kd_j}}{1-\mathbf{e}^{2\pi\mathbf{i}k}}$$

analytically continues to an entire function of k.

Proof. Let $\xi\in D$ be an element. Consider the real hyperplane $C_t=\mathbf{i}t\xi+\mathfrak{h}_{\mathbb{R}}$, $t>0$. Then C_t does not intersect reflection hyperplanes, so we have a continuous branch of $\delta(\mathbf{x})^{2k}$ on C_t which tends to the positive branch in D as $t\to 0$. Then, it is easy to see that for any $w\in W$, the limit of this branch in the chamber $w(D)$ will be $e^{2\pi\mathbf{i}k\ell(w)}|\delta(\mathbf{x})|^{2k}$, where $\ell(w)$ is the length of w. Therefore, by letting $t=0$, we get

$$(2\pi)^{-r/2}\int_{C_t}\mathbf{e}^{-(\mathbf{x},\mathbf{x})/2}\delta(\mathbf{x})^{2k}\mathrm{d}\mathbf{x}=\frac{1}{|W|}F(k)(\sum_{w\in W}e^{2\pi\mathbf{i}k\ell(w)})$$

(as this integral does not depend on t by Cauchy's theorem). But it is well known that

$$\sum_{w\in W}\mathbf{e}^{2\pi\mathbf{i}k\ell(w)}=\prod_{j=1}^{r}\frac{1-\mathbf{e}^{2\pi\mathbf{i}kd_j}}{1-\mathbf{e}^{2\pi\mathbf{i}k}},$$

([Hu], p.73), so

$$(2\pi)^{-r/2}|W|\int_{C_t}\mathbf{e}^{-(\mathbf{x},\mathbf{x})/2}\delta(\mathbf{x})^{2k}\mathrm{d}\mathbf{x}=G(k).$$

Since $\int_{C_t}\mathbf{e}^{-(\mathbf{x},\mathbf{x})/2}\delta(\mathbf{x})^{2k}\mathrm{d}\mathbf{x}$ is clearly an entire function, the statement is proved. $\square$

Corollary 4.13. *For every $k_0\in[-1,0]$ the total multiplicity of all the roots of b of the form k_0-p, $p\in\mathbb{Z}_+$, equals the number of ways to represent k_0 in the form $-m/d_i$, $m=1,\ldots,d_i-1$. In other words, the roots of b are $k_{i,m}=-m/d_i-p_{i,m}$, $1\leq m\leq d_i-1$, where $p_{i,m}\in\mathbb{Z}_+$.*

Proof. We have

$$G(k-p)=\frac{F(k)}{b(k-1)\cdots b(k-p)}\prod_{j=1}^{r}\frac{1-\mathbf{e}^{2\pi\mathbf{i}kd_j}}{1-\mathbf{e}^{2\pi\mathbf{i}k}},$$

Now plug in $k = 1 + k_0$ and a large positive integer p. Since by Proposition 4.12 the left hand side is regular, so must be the right hand side, which implies the claimed upper bound for the total multiplicity, as $F(1 + k_0) > 0$. The fact that the bound is actually attained follows from the fact that the polynomial b has degree exactly $|\mathcal{S}|$ (Proposition 4.11), and the fact that all roots of b are negative rational (Proposition 4.3). □

It remains to show that in fact in Corollary 4.13, $p_{i,m} = 0$ for all i, m; this would imply (ii) and hence (i).

Proposition 4.14. *Identity* (4.1) *of Theorem 4.1 is satisfied in* $\mathbb{C}[k]/k^2$.

Proof. Indeed, we clearly have $F(0) = 1$. Next, a rank 1 computation gives $F'(0) = -\gamma|\mathcal{S}|$, where γ is the Euler constant (i.e. $\gamma = \lim_{n\to+\infty}(1+\cdots+1/n - \log n)$), while the derivative of the right hand side of (4.1) at zero equals to

$$-\gamma \sum_{i=1}^{r}(d_i - 1).$$

But it is well known that

$$\sum_{i=1}^{r}(d_i - 1) = |\mathcal{S}|,$$

([Hu], p.62), which implies the result. □

Proposition 4.15. *Identity* (4.1) *of Theorem 4.1 is satisfied in* $\mathbb{C}[k]/k^3$.

Note that Proposition 4.15 immediately implies (ii), and hence the whole theorem. Indeed, it yields that

$$(\log F)''(0) = \sum_{i=1}^{r}\sum_{m=1}^{d_i-1}(\log\Gamma)''(m/d_i),$$

so by Corollary 4.13

$$\sum_{i=1}^{r}\sum_{m=1}^{d_i-1}(\log\Gamma)''(m/d_i + p_{i,m}) = \sum_{i=1}^{r}\sum_{m=1}^{d_i-1}(\log\Gamma)''(m/d_i),$$

which implies that $p_{i,m} = 0$ since $(\log\Gamma)''$ is strictly decreasing on $[0, \infty)$.

To prove Proposition 4.15, we will need the following result about finite Coxeter groups.

Let $\psi(W) = 3|\mathcal{S}|^2 - \sum_{i=1}^{r}(d_i^2 - 1)$.

Lemma 4.16. *One has*

$$\psi(W) = \sum_{G\in \mathrm{Par}_2(W)} \psi(G), \tag{4.4}$$

where $\mathrm{Par}_2(W)$ *is the set of parabolic subgroups of* W *of rank 2.*

Proof. Let

$$Q(q) = |W|\prod_{i=1}^{r}\frac{1-q}{1-q^{d_i}}.$$

It follows from Chevalley's theorem that

$$Q(q) = (1-q)^r \sum_{w\in W} \det(1-qw|_{\mathfrak{h}})^{-1}.$$

Let us subtract the terms for $w = 1$ and $w \in \mathcal{S}$ from both sides of this equation, divide both sides by $(q-1)^2$, and set $q = 1$ (cf. [Hu], p.62, formula (21)). Let W_2 be the set of elements of W that can be written as a product of two different reflections. Then by a straightforward computation we get

$$\frac{1}{24}\psi(W) = \sum_{w\in W_2} \frac{1}{r - \mathrm{Tr}\,_{\mathfrak{h}}(w)}.$$

In particular, this is true for rank 2 groups. The result follows, as any element $w \in W_2$ belongs to a unique parabolic subgroup G_w of rank 2 (namely, the stabilizer of a generic point $\mathfrak{h}^w$, [Hu], p.22). □

Proof of Proposition 4.15. Now we are ready to prove the proposition. By Proposition 4.14, it suffices to show the coincidence of the second derivatives of (4.1) at $k = 0$. The second derivative of the right hand side of (4.1) at zero is equal to

$$\frac{\pi^2}{6}\sum_{i=1}^{r}(d_i^2-1) + \gamma^2|\mathcal{S}|^2.$$

On the other hand, we have

$$F''(0) = (2\pi)^{-r/2}\sum_{\alpha,\beta\in\mathcal{S}}\int_{\mathfrak{h}_{\mathbb{R}}} \mathbf{e}^{-(\mathbf{x},\mathbf{x})/2}\log\alpha^2(\mathbf{x})\log\beta^2(\mathbf{x})\mathrm{d}\mathbf{x}.$$

Thus, from a rank 1 computation we see that our job is to establish the equality

$$(2\pi)^{-r/2}\sum_{\alpha\neq\beta\in\mathcal{S}}\int_{\mathfrak{h}_{\mathbb{R}}} \mathbf{e}^{-(\mathbf{x},\mathbf{x})/2}\log\alpha^2(\mathbf{x})\log\frac{\beta^2(\mathbf{x})}{\alpha^2(\mathbf{x})}\mathrm{d}\mathbf{x} = \frac{\pi^2}{6}\Big(\sum_{i=1}^{r}(d_i^2-1) - 3|\mathcal{S}|^2\Big) = -\frac{\pi^2}{6}\psi(W).$$

Since this equality holds in rank 2 (as in this case (4.1) reduces to the beta integral), in general it reduces to equation (4.4) (as for any $\alpha \neq \beta \in S$, s_α and s_β are contained in a unique parabolic subgroup of W of rank 2). The proposition is proved. □

4.3. **Application: the supports of $L_c(\mathbb{C})$.** In this subsection we will use the Macdonald-Mehta integral to computation of the support of the irreducible quotient of the polynoamial representation of a rational Cherednik algebra (with equal parameters). We will follow the paper [E3].

First note that the vector space $\mathfrak{h}$ has a stratification labeled by parabolic subgroups of W. Indeed, for a parabolic subgroup $W' \subset W$, let $\mathfrak{h}_{\mathrm{reg}}^{W'}$ be the set of points in $\mathfrak{h}$ whose stabilizer is W'. Then

$$\mathfrak{h} = \coprod_{W'\in \mathrm{Par}(W)} \mathfrak{h}_{\mathrm{reg}}^{W'},$$

where $\mathrm{Par}(W)$ is the set of parabolic subgroups in W.

For a finitely generated module M over $\mathbb{C}[\mathfrak{h}]$, denote the support of M by $\mathrm{supp}\,(M)$.

The following theorem is proved in [Gi1], Section 6 and in [BE] with different method. We will recall the proof from [BE] later.

Theorem 4.17. *Consider the stratification of* $\mathfrak{h}$ *with respect to stabilizers of points in* W*. Then the support* $\text{supp}\,(M)$ *of any object* M *of* $\mathcal{O}_c(W, \mathfrak{h})$ *in* $\mathfrak{h}$ *is a union of strata of this stratification.*

This makes one wonder which strata occur in $\text{supp}\,(L_c(\tau))$, for given c and τ. In [VV], Varagnolo and Vasserot gave a partial answer for $\tau = \mathbb{C}$. Namely, they determined (for W being a Weyl group) when $L_c(\mathbb{C})$ is finite dimensional, which is equivalent to $\text{supp}\,(L_c(\mathbb{C})) = 0$. For the proof (which is quite complicated), they used the geometry affine Springer fibers. Here we will give a different (and simpler) proof. In fact, we will prove a more general result.

Recall that for any Coxeter group W, we have the Poincaré polynomial:

$$P_W(q) = \sum_{w \in W} q^{\ell(w)} = \prod_{i=1}^{r} \frac{1 - q^{d_i(W)}}{1 - q}, \text{ where } d_i(W) \text{ are the degrees of } W.$$

Lemma 4.18. *If* $W' \subset W$ *is a parabolic subgroup of* W*, then* P_W *is divisible by* $P_{W'}$*.*

Proof. By Chevalley's theorem, $\mathbb{C}[\mathfrak{h}]$ is a free module over $\mathbb{C}[\mathfrak{h}]^W$ and $\mathbb{C}[\mathfrak{h}]^{W'}$ is a direct summand in this module. So $\mathbb{C}[\mathfrak{h}]^{W'}$ is a projective module, thus free (since it is graded).

Hence, there exists a polynomial $Q(q)$ such that we have

$$Q(q) h_{\mathbb{C}[\mathfrak{h}]^W}(q) = h_{\mathbb{C}[\mathfrak{h}]^{W'}}(q),$$

where $h_V(q)$ denotes the Hilbert series of a graded vector space V. Notice that we have $h_{\mathbb{C}[\mathfrak{h}]^W}(q) = \dfrac{1}{P_W(q)(1-q)^r}$, so we have

$$\frac{Q(q)}{P_W(q)} = \frac{1}{P_{W'}(q)}, \text{ i.e. } Q(q) = P_W(q)/P_{W'}(q).$$

□

Corollary 4.19. *If* $m \geq 2$ *then we have the following inequality:*

$$\#\{i | m \textit{ divides } d_i(W)\} \geq \#\{i | m \textit{ divides } d_i(W')\}.$$

Proof. This follows from Lemma 4.18 by looking at the roots of the polynomials P_W and $P_{W'}$. □

Our main result is the following theorem.

Theorem 4.20. [E3] *Let* $c \geq 0$*. Then* $a \in \text{supp}\,(L_c(\mathbb{C}))$ *if and only if*

$$\frac{P_W}{P_{W_a}}(\mathbf{e}^{2\pi \mathbf{i} c}) \neq 0.$$

We can obtain the following corollary easily.

Corollary 4.21.
(i) $L_c(\mathbb{C}) \neq M_c(\mathbb{C})$ *if and only if* $c \in \mathbb{Q}_{>0}$ *and the denominator* m *of* c *divides* d_i *for some* i*;*
(ii) $L_c(\mathbb{C})$ *is finite dimensional if and only if* $\dfrac{P_W}{P_{W'}}(\mathbf{e}^{2\pi \mathbf{i} c}) = 0$*, i.e., iff*

$$\#\{i | m \textit{ divides } d_i(W)\} > \#\{i | m \textit{ divides } d_i(W')\}.$$

for any maximal parabolic subgroup $W' \subset W$*.*

Remark 4.22. Varagnolo and Vasserot prove that $L_c(\mathbb{C})$ is finite dimensional if and only if there exists a regular elliptic element in W of order m. Case-by-case inspection shows that this condition is equivalent to the combinatorial condition of (2). Also, a uniform proof of this equivalence is given in the appendix to [E3], written by S. Griffeth.

Example 4.23. For type A_{n-1}, i.e., $W = \mathfrak{S}_n$, we get that $L_c(\mathbb{C})$ is finite dimensional if and only if the denominator of c is n. This agrees with our previous results in type A_{n-1}.

Example 4.24. Suppose W is the Coxeter group of type E_7. Then we have the following list of maximal parabolic subgroups and the degrees (note that E_7 itself is not a maximal parabolic).

Subgroups	E_7	D_6	$A_3 \times A_2 \times A_1$	A_6
Degrees	2,6,8,10,12,14,18	2,4,6,6,8,10	2,3,4,2,3,2	2,3,4,5,6,7
Subgroups	$A_4 \times A_2$	E_6	$D_5 \times A_1$	$A_5 \times A_1$
Degrees	2,3,4,5,2,3	2,5,6,8,9,12	2,4,5,6,8,2	2,3,4,5,6,2

So $L_c(\mathbb{C})$ is finite dimensional if and only if the denominator of c is $2, 6, 14, 18$.

The rest of the subsection is dedicated to the proof of Theorem 4.20. First we recall some basic facts about the Schwartz space and tempered distributions.

Let $\mathcal{S}(\mathbb{R}^n)$ be the set of Schwartz functions on $\mathbb{R}^n$, i.e.

$$\mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) | \forall \alpha, \beta, \sup |\mathbf{x}^\alpha \partial^\beta f(\mathbf{x})| < \infty\}.$$

This space has a natural topology.

A tempered distribution on $\mathbb{R}^n$ is a continuous linear functional on $\mathcal{S}(\mathbb{R}^n)$. Let $\mathcal{S}'(\mathbb{R}^n)$ denote the space of tempered distributions.

We will use the following well known lemma.

Lemma 4.25. (i) *$\mathbb{C}[\mathbf{x}]\mathbf{e}^{-\mathbf{x}^2/2} \subset \mathcal{S}(\mathbb{R}^n)$ is a dense subspace.*

(ii) *Any tempered distribution ξ has finite order, i.e., $\exists N = N(\xi)$ such that if $f \in \mathcal{S}(\mathbb{R}^n)$ satisfying $f = \mathrm{d}f = \cdots = \mathrm{d}^{N-1}f = 0$ on $\operatorname{supp}\xi$, then $\langle \xi, f\rangle = 0$.*

Proof of Theorem 4.20. Recall that on $M_c(\mathbb{C})$, we have the Gaussian form γ_c from Section 4.2. We have for $\mathrm{Re}\,(c) \le 0$,

$$\gamma_c(P,Q) = \frac{(2\pi)^{-r/2}}{F_W(-c)} \int_{\mathfrak{h}_\mathbb{R}} \mathbf{e}^{-\mathbf{x}^2/2} |\delta(\mathbf{x})|^{-2c} P(\mathbf{x})Q(\mathbf{x})\mathrm{d}\mathbf{x},$$

where P, Q are polynomials and

$$F_W(k) = (2\pi)^{-r/2} \int_{\mathfrak{h}_\mathbb{R}} \mathbf{e}^{-\mathbf{x}^2/2} |\delta(\mathbf{x})|^{2k} \mathrm{d}\mathbf{x}$$

is the Macdonald-Mehta integral.

Consider the distribution:

$$\xi_c^W = \frac{(2\pi)^{-r/2}}{F_W(-c)} |\delta(\mathbf{x})|^{-2c}.$$

It is well-known that this distribution is meromorphic in c (Bernstein's theorem). Moreover, since $\gamma_c(P,Q)$ is a polynomial in c for any P and Q, this distribution is in fact holomorphic in $c \in \mathbb{C}$.

Proposition 4.26.

$$\begin{aligned}\operatorname{supp}(\xi_c^W) &= \{a \in \mathfrak{h}_{\mathbb{R}} | \frac{F_{W_a}}{F_W}(-c) \neq 0\} = \{a \in \mathfrak{h}_{\mathbb{R}} | \frac{P_W}{P_{W_a}}(e^{2\pi \mathrm{i} c}) \neq 0\} \\ &= \{a \in \mathfrak{h}_{\mathbb{R}} | \#\{i | \textit{denominator of } c \textit{ divides } d_i(W)\} \\ &\qquad = \#\{i | \textit{denominator of } c \textit{ divides } d_i(W_a)\}\}.\end{aligned}$$

Proof. First note that the last equality follows from the product formula for the Poincaré polynomial, and the second equality from the Macdonald-Mehta identity. Now let us prove the first equality.

Look at ξ_c^W near $a \in \mathfrak{h}$. Equivalently, we can consider

$$\xi_c^W(\mathbf{x}+a) = \frac{(2\pi)^{-r/2}}{F_W(-c)} |\delta(\mathbf{x}+a)|^{-2c}$$

with $\mathbf{x}$ near 0. We have

$$\begin{aligned}\delta_W(\mathbf{x}+a) &= \prod_{s \in \mathcal{S}} \alpha_s(\mathbf{x}+a) = \prod_{s \in \mathcal{S}} (\alpha_s(\mathbf{x}) + \alpha_s(a)) \\ &= \prod_{s \in \mathcal{S} \cap W_a} \alpha_s(\mathbf{x}) \cdot \prod_{s \in \mathcal{S} \setminus \mathcal{S} \cap W_a} (\alpha_s(\mathbf{x}) + \alpha_s(a)) \\ &= \delta_{W_a}(\mathbf{x}) \cdot \Psi(\mathbf{x}),\end{aligned}$$

where Ψ is a nonvanishing function near a (since $\alpha_s(a) \neq 0$ if $s \notin \mathcal{S} \cap W_a$).

So near a, we have

$$\xi_c^W(\mathbf{x}+a) = \frac{F_{W_a}}{F_W}(-c) \cdot \xi_c^{W_a}(\mathbf{x}) \cdot |\Psi|^{-2c},$$

and the last factor is well defined since Ψ is nonvanishing. Thus $\xi_c^W(\mathbf{x})$ is nonzero near a if and only if $\frac{F_{W_a}}{F_W}(-c) \neq 0$ which finishes the proof. ☐

Proposition 4.27. *For $c \geq 0$,*

$$\operatorname{supp}(\xi_c^W) = \operatorname{supp} L_c(\mathbb{C})_{\mathbb{R}},$$

where the right hand side stands for the real points of the support.

Proof. Let $a \notin \operatorname{supp} L_c(\mathbb{C})$ and assume $a \in \operatorname{supp} \xi_c^W$. Then we can find a $P \in J_c(\mathbb{C}) = \ker \gamma_c$ such that $P(a) \neq 0$. Pick a compactly supported test function $\phi \in C_c^\infty(\mathfrak{h}_{\mathbb{R}})$ such that P does not vanish anywhere on $\operatorname{supp} \phi$, and $\langle \xi_c^W, \phi \rangle \neq 0$ (this can be done since $P(a) \neq 0$ and ξ_c^W is nonzero near a). Then we have $\phi/P \in \mathcal{S}(\mathfrak{h}_{\mathbb{R}})$. Thus from Lemma 4.25 (i) it follows that there exists a sequence of polynomials P_n such that

$$P_n(\mathbf{x})\mathbf{e}^{-\mathbf{x}^2/2} \to \frac{\phi}{P} \text{ in } \mathcal{S}(\mathfrak{h}_{\mathbb{R}}), \text{ when } n \to \infty.$$

So $PP_n\mathbf{e}^{-\mathbf{x}^2/2} \to \phi$ in $\mathcal{S}(\mathfrak{h}_{\mathbb{R}})$, when $n \to \infty$.

But we have $\langle \xi_c^W, PP_n\mathbf{e}^{-\mathbf{x}^2/2} \rangle = \gamma_c(P, P_n) = 0$ which is a contradiction. This implies that $\operatorname{supp} \xi_c^W \subset (\operatorname{supp} L_c(\mathbb{C}))_{\mathbb{R}}$.

To show the opposite inclusion, let P be a polynomial on $\mathfrak{h}$ which vanishes identically on $\operatorname{supp} \xi_c^W$. By Lemma 4.25 (ii), there exists N such that $\langle \xi_c^W, P^N(\mathbf{x})Q(\mathbf{x})\mathbf{e}^{-\mathbf{x}^2/2} \rangle = 0$. Thus, for any polynomial Q, $\gamma_c(P^N, Q) = 0$, i.e. $P^N \in \operatorname{Ker} \gamma_c$. Thus, $P|_{\operatorname{supp} L_c(\mathbb{C})} = 0$. This implies the required inclusion, since $\operatorname{supp} \xi_c^W$ is a union of strata. □

Theorem 4.20 follows from Proposition 4.26 and Proposition 4.27. □

4.4. **Notes.** Our exposition in Sections 4.1 and 4.2 follows the paper [E2]; Section 4.3 follows the paper [E3].

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