18.727 Topics in Algebraic Geometry: Algebraic Surfaces
Spring 2008

ALGEBRAIC SURFACES, LECTURE 21

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From last time: $f : X \to B$ is an elliptic/quasi-elliptic fibration, $F_{b_i} = m_i P_i$ multiple fibers, $R^1 f_* \mathcal{O}_X = L \oplus T$, for L invertible on B and T torsion. $b \in \operatorname{Supp}(T) \Leftrightarrow h^0(\mathcal{O}_{F_b}) \geq 0 \Leftrightarrow h^1(\mathcal{O}_{F_b}) \geq 2 \Leftrightarrow F_b$ is an exceptional/wild fiber.

Theorem 1. *With the above notation, $\omega_X = f^*(L^{-1} \otimes \omega_B) \otimes \mathcal{O}_X(\sum a_i P_i)$, where $0 \leq a_i < m, a_i = m_i - 1$ unless F_{b_i} is exceptional, and $\deg(L^{-1} \otimes \omega_B) = 2p_a(B) - 2 + \chi(\mathcal{O}_X) + \ell(T)$, where $\ell(T)$ is its length as an $\mathcal{O}_B$-module.*

Proof. We have proved most of this: specifically, we have that $\omega_X = f^*(L^{-1} \otimes \omega_B) \otimes \mathcal{O}_X(\sum a_i P_i)$ for $0 \leq a_i < m$. We have a Leray spectral sequence $E_2^{pq} = H^p(B, R^q f_* \mathcal{O}_X) \implies H^{p+q}(X, \mathcal{O}_X)$. The smaller order terms give us a short exact sequence

$$\begin{aligned} 0 \to H^0(\mathcal{O}_B) \to H^1(\mathcal{O}_X) \to H^0(R^1 f_* \mathcal{O}_X) \to H^2(\mathcal{O}_B) = 0 \\ 0 \to H^2(\mathcal{O}_X) \to H^1(R^1 f_* \mathcal{O}_X) \to 0 \end{aligned} \tag{1}$$

Using this, we see that

$$\begin{aligned} \chi(\mathcal{O}_X) &= h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X) \\ &= h^0(\mathcal{O}_B) - h^1(\mathcal{O}_B) - h^0(L \oplus T) + h^1(L \oplus T) \\ &= \chi(\mathcal{O}_B) - \chi(L) - h^0(T) \\ &= -\deg L - \ell(T) \end{aligned} \tag{2}$$

by Riemann-Roch, so $\deg L = -\chi(\mathcal{O}_X) - \ell(T)$. Since $\deg \omega_B = 2p_a(B) - 2$, we have $\deg(L^{-1} \otimes \omega_B) = 2p_a(B) - 2 + \chi(\mathcal{O}_X) + \ell(T)$. It remains to show that $a_i = m_i - 1$ if F_{b_i} is not exceptional. If fact, we can prove something stronger: let α_i be the order of $\mathcal{O}_X(P_i) \otimes \mathcal{O}_{P_i}$ in $\operatorname{Pic}(P_i)$. Then we claim that

1. α_i divides m_i and $a_i + 1$,
2. $h^0(P_i, \mathcal{O}_{(\alpha_i+1)P_i}) \geq 2$ and $h^0(P_i, \mathcal{O}_{\alpha_i P_i}) = 1$, and
3. $h^0(P_i, nP_i)$ is a nondecreasing function of n.

Assuming this, if $a_i < m_i - 1$, then $\alpha_i < m_i$, so $m_i P_i$ is exceptional by (b) and (c), since then $h^0(\mathcal{O}_{m_i P_i}) \geq 2$.

We now prove the claim. If $m > n \geq 1$, then $\mathcal{O}_{mP} \to \mathcal{O}_{nP} \to 0$ gives $H^1(P, \mathcal{O}_{mP}) \to H^1(P, \mathcal{O}_{nP}) \to 0$, implying that $n \mapsto h^1(P, \mathcal{O}_{nP})$ is nondecreasing. But by Riemann-Roch and the definition of canonical type, $\chi(\mathcal{O}_{nP}) = 0$, so

$h^0 = h^1$ is also nondecreasing. Now, by the definition of α_i, $\mathcal{O}_X(\alpha_i P_i) \otimes \mathcal{O}_{P_i} \cong \mathcal{O}_{P_i}$, implying that $\mathcal{O}_X(-n_i P_i) \otimes \mathcal{O}_{P_i} \cong \mathcal{O}_{P_i}$ as well. We thus obtain an exact sequence $0 \to \mathcal{O}_X(-\alpha_i P_i) \otimes \mathcal{O}_{P_i} = \mathcal{O}_{P_i} \to \mathcal{O}_{(\alpha_i+1)P_i} \to \mathcal{O}_{\alpha_i P_i} \to 0$, inducing a long exact sequence

$$0 \to k \cong H^0(\mathcal{O}_{P_i}) \to H^0(\mathcal{O}_{(\alpha_i+1)P_i}) \to H^0(\mathcal{O}_{\alpha_i P_i}) \to \cdots \tag{3}$$

and $h^0(\mathcal{O}_{(\alpha_i+1)P_i}) \geq 2$. But for $1 \leq j < \alpha_i, L_j = \mathcal{O}_X(-jP_i) \otimes \mathcal{O}_{P_i}$ is an invertible $\mathcal{O}_{P_i}$-module whose degree in each component of P_i equals 0. Since $L_j \ncong \mathcal{O}_{P_i}, H^0(L_j) = 0$, and $0 \to L_j \to \mathcal{O}_{(j+1)P_i} \to \mathcal{O}_{jP_i} \to 0$ gives $H^0(\mathcal{O}_{(j+1)P_i}) \cong H^0(\mathcal{O}_{jP_i})$. Since $H^0(\mathcal{O}_P) \cong k$ for P icoct, $H^0(\mathcal{O}_{2P}) \cong \cdots \cong H^0(\mathcal{O}_{\alpha P}) \cong k$ as well.

Finally,

$$(\mathcal{O}_X(P_i) \otimes \mathcal{O}_{P_i})^{m_i} \cong \mathcal{O}_X(F_{b_i}) \otimes \mathcal{O}_{P_i} \cong \mathcal{O}_{P_i} \tag{4}$$

This is proved as follows, Since the fiber is cut out by a rational function f, $H^0(\mathcal{O}_X(F_{b_i}) \otimes \mathcal{O}_{P_i}) \neq 0$. Via the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(F_{b_i}) \overset{1/f \mapsto \overline{1/f}}{\to} \mathcal{O}_X(F_{b_i}) \otimes \mathcal{O}_{F_{b_i}} \to 0 \tag{5}$$

we get a global section of $\mathcal{O}_X(F_{b_i}) \otimes \mathcal{O}_{F_{b_i}}$. But this also has degree 0 along the components. So it must be trivial, but what we proved for icoct. We also have

$$\begin{aligned} \mathcal{O}_X((a_i+1)P_i) \otimes \mathcal{O}_{P_i} &\cong \omega_X \otimes \mathcal{O}_X(P_i) \otimes \mathcal{O}_{P_i} \\ &\cong \omega_{P_i} \cong \mathcal{O}_{P_i} \end{aligned} \tag{6}$$

implying that $\alpha_i | a_i + 1$ as desired. ∎

Corollary 1. $K^2 = 0$.

Corollary 2. *If $h^1(\mathcal{O}_X) \leq 1$, then either $a_i + 1 = m_i$ or $a_i + \alpha_i + 1 = m_i$.*

Proof. Exercise. ∎

Remark. Raynaud showed that m_i/α_i is a power of $p = \mathrm{char}(k)$ (or is 1 if $\mathrm{char}(k) = 0$). Therefore, there are no exceptional fibers in characteristic 0.

1. Classification (contd.)

If $f : X \to B$ is an elliptic/quasi-elliptic fibration, then

$$\omega_X = f^*(L^{-1} \otimes \omega_B) \otimes \mathcal{O}_X(\sum a_i P_i), 0 \leq a_i < m_i \tag{7}$$

If $n \geq 1$ is a multiple of $m_1, \ldots, m_r$, then

$$H^0(X, \omega_X^{\otimes n}) = H^0(B, L^{-n} \otimes \omega_B^n \otimes \mathcal{O}_B(\sum a_i(n/m_i)b_i)) \tag{8}$$

Now we recall the 4 classes of surfaces:

(a) $\exists$ an integral curve C on X s.t. $K \cdot C < 0$.

(b) $K \equiv 0$.
(c) $K^2 = 0$, $K \cdot C \geq 0$ for all integral curves C, and $\exists C'$ s.t. $K \cdot C' > 0$.
(d) $K^2 > 0$, and $K \cdot C \geq 0$ for all integral curves C.

Lemma 1. *If X is in (a), then $\kappa(X) = -\infty$, i.e. $p_n = 0$ for all $n \geq 1$. If X is in (b), then $\kappa(X) \leq 0$. If X has an elliptic or quasielliptic fibration $f : X \to B$, and if we let $\lambda(f) = 2p_a(B) - 2 + \chi(\mathcal{O}_X) + \ell(T) + \sum \frac{a_i}{m_i}$, then X is not in class (d) and*

- *X is in (a) iff $\lambda(f) < 0$, in which case $\kappa(X) = -\infty$,*
- *X is in (b) iff $\lambda(f) = 0$, in which case $\kappa(X) = 0$,*
- *X is in (c) iff $\lambda(f) > 0$, in which case $\kappa(X) = 1$.*

Proof. If $K \cdot C < 0$, then X is ruled, and $\kappa(X) = \infty$. We did this before, and there is an easy way to see that $p_n = 0$ for all $n \geq 1$. For every divisor $D \in \mathrm{Div}\,(X), \exists n_D$ s.t. $|D + nK| = \varnothing$ for $n > n_D$. (Since $(D + nK) \cdot C = D \cdot C + n(K \cdot C)$ becomes negative eventually. Now C is effective. We claim that $C^2 \geq 0$, so by our useful lemma, $6\,|D + nK|$ can't have an effective divisor. If $C^2 < 0$, then $C \cdot K < 0$ would imply that C was an exceptional curve of the first kind, contradicting the minimality of X. Thus, $C^2 \geq 0$.) In particular, $D = K$ gives $|nK| = \varnothing$ for large enough n, implying that $|nK| = \varnothing$ for all n (since $p_n < p_{mn}$).

Next, assume $K \equiv 0$ (case (b)). If $p_n \geq 2$, then $\dim |nK| \geq 1 \implies \exists$ a strictly positive divisor $\Delta > 0$ in $|nK|$. Then $\Delta \cdot H > 0$ for a hypersurface section, contradicting $nK \cdot H = 0$ since $K \equiv 0$. So $p_n \leq 1$ for all n, implying that $\kappa(X) \leq 0$.

Now assume X has an elliptic/quasielliptic fibration, and let $M = f_*(\omega_X) = L^{-1} \otimes \omega_B$ from last time. Then M has degree $\lambda(f)$. Let H be a very ample divisor on X. Then $\pi = f|_H : H \to B$ is some finite map of degree $= H \cdot F > 0$. Now $n(K \cdot H) = \deg\,(\omega_X^n|_H) = \deg_H(\pi^* M) = (\deg \pi)(\deg_B M) = (H \cdot F)\lambda_f$. So if $\lambda_f < 0$, then $K \cdot H < 0$ and X is in (a).

Similarly, $\lambda(f) = 0 \implies K \cdot H = 0$ for every irreducible hyperplane section H, and any curve C can be written, up to $\sim$, as the difference of 2 such. This implies that $K \cdot C = 0\,\forall\, C \implies K \equiv 0$. Lastly, $\lambda(f) > 0 \implies K \cdot C > 0$ for all horizontal irreducible C. For vertical C, $K \cdot C = 0$ by the formula for K, implying that $K \cdot C \geq 0$ for all C integral, $(K^2) = 0$ by the formula, implying that we are in class (c). ☐

Let X be a minimal surface with $K^2 = 0, p_g \leq 1$ (in particular, every surface in class (b) is of this form. Then Noether's formula gives $10 - 8q + 12p_g = b_2 + 2\Delta$. Since $p_g \leq 1, 0 \leq \Delta \leq 2p_g \leq 2$, also $\Delta = 2(q - s)$ is even, we obtain the following possibilities.

(1) $b_2 = 22, b_1 = 0, \chi(\mathcal{O}_X) = 2, q = 0, p_g = 1, \Delta = 0$.
(2) $b_2 = 14, b_1 = 2, \chi(\mathcal{O}_X) = 1, q = 1, p_g = 1, \Delta = 0$.

(3) $b_2 = 10, b_1 = 0, \chi(\mathcal{O}_X) = 1$, and either $q = 0, p_g = 0, \Delta = 0$ or $q = 1, p_g = 1, \Delta = 2$.
(4) $b_2 = 6, b_1 = 4, \chi(\mathcal{O}_X) = 0, q = 2, p_g = 1, \Delta = 0$.
(5) $b_2 = 2, b_1 = 2, \chi(\mathcal{O}_X) = 0$, and either $q = 1, p_g = 1, \Delta = 0$ or $q = 2, p_g = 0, \Delta = 2$.

Note. If X is in class (b) and $p_g = 1$, then $K \sim 0$ (because $K = 0, H^0(K) \neq 0$ imply that $K \sim 0$.).

Let's deal with case 4 of class (b).

Proposition 1. *Let X be minimal in class (b), and $b_2 = 2, b_1 = 2$. Then $s = 1$, $\mathrm{Alb}\,(X)$ is an elliptic curve, and $X \to \mathrm{Alb}\,(X)$ gives an elliptic/quasielliptic fibration.*

Proof. Let's see that the fibers of f are irreducible. If not, we would have $\rho > 2(F, H, \text{component of } F)$ and $b_2 \geq \rho > 2$, contradicting $b_2 = 2$. Now, to see that the fibers are not multiple, note that $\chi(\mathcal{O}_X) = 0$ from the list.

$$\deg\,(L^{-1} \otimes \omega_B) = 2p_a(B) - 2 + \chi(\mathcal{O}_X) + \ell(T) = \ell(T) \geq 0 \tag{9}$$

Since $\omega_X = f^*(L^{-1} \otimes \omega_B) \otimes \mathcal{O}_X(\sum a_i P_i) \equiv 0$, we see that $\ell(T) \cdot f^{-1}(y) + \sum a_i P_i \equiv 0$. But it is an effective divisor, implying that all the $a_i = 0, \ell(T) = 0$ and thus $a_i = m_i - 1 \,\forall\, i$ (there are no wild fibers since $T = 0$). So $m_i = 1 \,\forall\, i$. Thus, we have integral fibers, which is the case of a bielliptic surface. $\square$