18.727 Topics in Algebraic Geometry: Algebraic Surfaces
Spring 2008

ALGEBRAIC SURFACES, LECTURE 15

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1. Elliptic Surfaces (contd.)

Assume we are over $\mathbb{C}$. Given $f : X \to C$, we can associate the functional invariant $j : C \to \mathbb{P}^1, c \mapsto j(f_c)$ and a homological invariant: let $c_1, \ldots, c_n$ be points over which the fibers are singular, $C^* = C \setminus \{c_1, \ldots, c_n\}$ and $X^* = f^{-1}(C^*)$; then we have the sheaf $R^1 f_* \mathbb{Z}$, which is a homological invariant; $(R^1 f_* \mathbb{Z}) \otimes_C C = H^1(F_{\mathbb{C}}, \mathbb{Z})$, so it is a locally constant sheaf. We get a representation $G : \pi_1(C^*) \to \mathrm{SL}_2(\mathbb{Z})$ called the global monodromy. The equivalence class of this representation is called the *homological invariant*. The local monodromy at a point c_i is the image of a loop around that point in $\mathrm{SL}_2(\mathbb{Z})$. We can write down the conjugacy class of the local monodromy for the bad fibers: for a I_n fiber, it is $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. J and G determine the elliptic fibration up to isomorphism. Given any elliptic fibration, we get a corresponding Jacobian fibration which does have a section $O : C \to X$.

Let $X \to C$ be an elliptic fibration with section), and let the class of the fiber be F. For each singular fiber, let $F_v = \Theta_{v,0} + \sum M_{v,i} \Theta_{v,i}$, where $\Theta_{v,0}$ is the identity component. Algebraic and numerical equivalence are the same for elliptic surfaces, and $K_X = f^*(K_C - N)$, where N is isomorphic to the normal bundle of the zero section O in X. Note that $\mathcal{O}_C(N) = R^1 f_* \mathcal{O}_X$. Thus, $K_S \equiv (2g-2+\chi)F$, where χ = the "arithmetic genus" of $X = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X)$, with $(K_S)^2 = 0$ and $\deg N = 0^2 = -\chi$. For any section P we have

$$2g - 2 = P(P + K) = P^2 + P \cdot K = P^2 + 2g - 2 + \chi \tag{1}$$

so $P^2 = -\chi$.

Let T be the subgroup of $NS(X)$ generated by the zero section $\mathcal{O}$ and all irreducible components of fibers (called the trivial lattice)

$$X(C) \cong E(K) \cong NS(X)/T \tag{2}$$

where $K = k(C)$. Here T is torsion free, and the map is given by $P \in E(K) \mapsto P \bmod T$. We can obtain the Shioda-Tate formula: rk $NS(X) = 2 + \sum(m_v - 1) +$ rk , $E(k)$, where rk $E(k)$ is the Mordell-Weil rank.

Remark. An open arithmetic question: is the rank of $E(K)$ unbounded for $K = K(C)$, the function field of a curve (e.g. $C = \mathbb{P}^1$, $E(\mathbb{Q}(x))$)? For $\mathbb{F}_p(k)$, the answer is yes (Shafarevich-Tate). For $E(\mathbb{Q})$, we can write down a finite list of the possible torsion (Mazur's theorem).

2. Kodaira dimension 0

If X is a surface with $\kappa(X) = 0$, then $(K^2) = 0, p_g \leq 1$ (since $K^2 < 0$ would imply K ruled): we will show this later. Noether's formula says that $10 - 8q + 12p_g = b_2 + 2\Delta$, where $\Delta = 2(q - s)$. Then $0 \leq \Delta \leq 2p_g$ and Δ even implies that it is 0 or 2. Now, the four possibilities based on combinations of (b_1, b_2) are:

- $(0, 10)$ are Enriques surfaces, "classical" if $(p_g, q, \Delta) = (0, 0, 0)$ and "non-classical" if $(p_g, q, \Delta) = (1, 1, 2)$.
- $(2, 2)$ are bielliptic or hyperelliptic surfaces if $(p_g, q, \Delta) = (0, 1, 0)$, and quasi-hyperelliptic surfaces if $(p_g, q, \Delta) = (1, 2, 2)$.
- $(0, 22)$ are $K3$ surfaces, with $\chi(\mathcal{O}_X) = 0, q = 0, p_g = 1, \Delta = 0$, and
- $(4, 6)$ are abelian surfaces, with $\chi(\mathcal{O}_X) = 0, q = 2, p_g = 1, \Delta = 0$.

Note that, by the above formulae, there is another possible combination: $(b_1, b_2) = (2, 14), q_1, p_g = 1, \Delta = 0$.

Proposition 1. *No surface exists with this combination of invariants.*

Proof. Since $b_1 = 2 = 2s$, the Picard variety of X has dimension 1. So $\exists$ a line bundle $\mathcal{L}_0$ on X difference from $\mathcal{O}_X$ but algebraically equivalent to 0. Since $K \equiv 0$ and $p_g = 1$, we get $K_X \sim \mathcal{O}_X$. Applying Riemann-Roch to $\mathcal{L}$ gives $\chi(\mathcal{L}) = \frac{1}{2}\mathcal{L} \cdot (\mathcal{L} \otimes K^{-1}) + \chi(\mathcal{O}_X) = \chi(\mathcal{O}_X) = 1$. Thus, $H^0(\mathcal{L} > 0$ or $H^0(\mathcal{L}^{-1}) = H^2(\mathcal{L}) > 0$. But $\mathcal{L} \equiv 0$ gives $\mathcal{L} \sim \mathcal{O}_X$, a contradiction. $\square$

2.1. **Abelian surfaces.**

These are smooth, complete group varieties of dimension 2 over k. For the general theory, we refer to Mumford's Abelian Varieties and Birkenhaake-Lange's Complex Abelian Varieties. Over $\mathbb{C}$, these are complex tori: $\mathbb{C}^2/\Lambda$, equipped with a Riemann form $\psi : \Lambda \times \Lambda \to \mathbb{Z}$. This form is alternating, bilinear, and satisfies the following two conditions:

- $\psi_{\mathbb{R}} : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{R}$ satisfies $\psi_{\mathbb{R}}(iv, iw) = \psi_R(v, w)$ for $v, w \in \mathbb{C} \times \mathbb{C}$.
- The associated Hermitian form $H(v, w) = \psi_{\mathbb{R}}(iv, w) + i\psi_{\mathbb{R}}(v, w)$ is positive definite.

This gives rise to a line bundle $\mathcal{L}$ on $\mathbb{C}^2/\Lambda$ which is the class of an ample line bundle. $\mathcal{L}$ is called a polarization of the abelian variety, and

$$h^0(\mathcal{L}) = \sqrt{\operatorname{disc}(\phi)} = |\operatorname{Pf}(\psi)| \tag{3}$$

In general, over any characteristic, $A^\vee = \operatorname{Pic}^0 A$ is the dual abelian variety, and it is automatically reduced. A polarization is an isogeny $A \to A^\vee$: for example, whenever we have an ample line bundle $\mathcal{L}$ on A, the subgroup $K(\mathcal{L}) = \{x \in A \mid T_x^*\mathcal{L} \cong \mathcal{L}\}$ is finite. The map $A \to A^\vee, x \mapsto T_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$ gives an isogeny with finite kernel $K(L)$, i.e. a polarization. For a line bundle $\mathcal{L}$,

$$\chi(\mathcal{L}) = \frac{\mathcal{L}^g}{\dim A} = \frac{\mathcal{L}^2}{2}, \chi(\mathcal{L})^2 = \deg \phi_L \tag{4}$$

The polarization is principal if it has degree 1. There are two common examples of abelian surfaces with principal polarization:

(1) $J(C)$, C a genus 2 curve. Here, we can explicitly write down equations for $J(C)$ (e.g. Cassels-Flynn, Prolegomena to a Middlebrow Arithmetic of Curves of Genus 2). The image of C is $J(C)$ under $x \mapsto [x] - [x_0]$, where $x_0 \in C$ is a fixed point, is the *theta divisor* and it generates $NS(J(C))$ for a generic C. We have $C^2 = C \cdot (C + K) = 2g - 2 = 2$ since $K \equiv 0$ for an abelian variety. The theta divisor gives the principal polarization $\chi(\mathcal{O}_X(C)) = \frac{C^2}{2} = 1$, so $\deg \phi_C = \chi(\mathcal{O}_X(C))^2 = 1$.

(2) Fix E_2 with E_1, E_2 elliptic curves. Then $E_1 + E_2$ gives a principally polarized abelian variety, with $E_1 \cdot E_2 = 1, E_i^2 = 0 \implies (E_1 + E_2)^2/2 = 1$.

The moduli space of principally polarized abelian varieties is made up of these two types of points, with the second type forming the boundary.

There are lots of arithmetic questions here. By the Mordell-Weil theorem, if A is defined over a number field K, then $A(K)$ is finitely generated for a number field L containing K. What is the structure of the torsion subgroup, and what can you say about the rank? Merel showed that, given a positive integer d, there is a constant B_d, depending only on d, s.t. for any number field K of degree d over $\mathbb{Q}$, and any elliptic curve E/K with a point of order n, $n \leq B_d$. This also bounds the size of $E(K)_{\text{tors}}$. This is called the strong uniform boundedness conjecture. This is a broad generalization of Mazur and Kamienny's work. The analogue for abelian surfaces is not known. Also, we don't know much about ranks. The Galois representations that arise are of course very important. For instance, considering the action of $\mathrm{Gal}(\overline{K}/K)$ on $A[N]$ for N prime gives a map to the symplectic group $\mathrm{Sp}_4(\mathbb{F}_N)$.

For an abelian surface, $H^2_{\text{ét}}(A, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell^6$ with symplectic form induced from $H^2(A, \mathbb{Z}) \cong U^3, U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. NS has rank between 1 and 4 over $\mathbb{C}$ (because of Hodge theory), e.g. $A = J(C)$ ($\mathrm{rk}\,\mathrm{NS} = 1$, C generic), $A = E_1 \times E_2$ ($\mathrm{rk}\,\mathrm{NS} = 2$, E_1 not isogenous to E_2), $A = E_1 \times E_1$ ($\mathrm{rk}\,\mathrm{NS} = 3$, E_1 not CM), and $A = E_1 \times E_1$ ($\mathrm{rk}\,\mathrm{NS} = 4$, E_1 CM). In characteristic p, we can also have rank 6 for $E_1 \times E_1$, E_1 a supersingular elliptic curve.

2.2. $K3$ **surfaces.** These have $b_2 = 22, b_1 = 0, \chi(\mathcal{O}_X) = 2, q = 0, p_g = 1, \Delta = 0$. This is equivalent to saying $K_X \cong \mathcal{O}_X$ and $q = 0$. (exercise).

Proposition 2. $\mathrm{Pic}^{\tau}(X) = 0$, *i.e.* $\mathrm{Pic}(X)$ *is torsion free, i.e. numerical, algebraic, and linear equivalence are the same, implying that* $\mathrm{Pic}(X)$ *is a free abelian group of finite rank. Moreover, if* $f : X' \to X$ *is a connected étale cover of* X*, then* f *is an isomorphism, i.e.* X *is algebraically simply connected.*

Proof. If L is s.t. $L \equiv 0$ then, by Riemann-Roch, we get (since $K \sim 0 \implies K \equiv 0$)

$$\chi(L) = \chi(\mathcal{O}_X) + \frac{1}{2}L \cdot (L + K) = \chi(\mathcal{O}_X) = 2 \implies h^0(L) \neq 0 \tag{5}$$

or $H^2(L) = H^0(L^{-1})) \neq 0$: since $L \equiv 0$, we must have $L \cong \mathcal{O}_X$ or

$$L^{-1} \cong \mathcal{O}_X \implies L \cong \mathcal{O}_X \implies \mathrm{Pic}^{\tau} = 0 \tag{6}$$

[let $D \in H^0(L)$: if D were effective then $L \cdot H = D \cdot H > 0$ for H ample]. For the second part, let $f : X' \to X$ be étale of degree n. $\omega_{X'} = f^*(\omega_X) \cong \mathcal{O}_{X'}$ from

$$0 \to f^*(\Omega^1_{X/k}) \to \Omega^2_{X/k} \to \Omega^1_{X'/X} = 0 \tag{7}$$

. So $\kappa(X') = 0$, X' is a minimal surface, and $p_g(X') = 1 \implies X'$ is in in the list of possibilities for Kodaira dimension 0. Also $\chi(\mathcal{O}_{X'}) = n\chi(\mathcal{O}_X) = 2n$. But no surface in the list has $\chi(\mathcal{O}_{X'}) > 2 \implies n = 1$. □