18.727 Topics in Algebraic Geometry: Algebraic Surfaces
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ALGEBRAIC SURFACES, LECTURE 12

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Today we will prove the uniqueness of minimal models of non-ruled surfaces (in characteristic 0) and talk about the characterization of ruled surfaces.

Theorem 1 (Grothendieck-Cartier)**.** *In characteristic* 0*, a group scheme* G *is always reduced.*

Proposition 1. *Let* X *be a surface,* $\alpha : X \to \mathrm{Alb}\,(X)$ *the Albanese map. Suppose* $\alpha(X)$ *is a curve* C*. Then* C *is a smooth curve of genus* q*, and the fibers of* α *are connected.*

Lemma 1. *Suppose* α *factors as* $X \xrightarrow{f} T \xrightarrow{j} \mathrm{Alb}\,(X)$ *with* f *surjective. Then* $\tilde{j} : \mathrm{Alb}\,(T) \to \mathrm{Alb}\,(X)$ *is an isomorphism.*

Lemma 2. *Let* X *be a surface with* $p_g = 0, q \geq 1, \alpha : X \to \mathrm{Alb}\,(X)$ *its Albanese map. Then* $\alpha(X)$ *is a curve.*

Proof. If $Y = \alpha(X)$ is a surface, then the morphism $\alpha' : X \to Y$ is generically finite, hence generically étale (in characteristic 0). Pick a smooth point $y \in Y$, and find an invariant differential form $\omega : H^0(\mathrm{Alb}\,(X), \Omega^2)$ which is nonzero at y (since $\mathrm{Alb}\,(X)$ is an abelian variety). Then $\alpha^*\omega$ is a nonzero element of $H^0(X, \omega_X)$, contradicting $p_g = 0$. □

Theorem 2. *Let* X, X' *be two nonruled minimal surfaces. Then every birational map from* X *to* X' *is an isomorphism. In particular, every nonruled surface admits a unique minimal model up to isomorphism. The group of birational maps from a nonruled minimal surface to itself coincides with the group of automorphisms of the surface.*

Proof. (In characteristic 0: holds in positive characteristic with some modifications.) Let $\phi : X' \dashrightarrow X$ be a birational map. Then $\exists$ a series of blowups $\pi_1 \circ \cdots \circ \pi_n : \tilde{X} \to X$ resolving ϕ to a morphism $f : \tilde{X} \to X$. Choose one with n minimal. If $n = 0$, we are done, so assume that $n \geq 1$. Let E be the exceptional curve of the blowup π_n. Then $f(E)$ is a curve in X, otherwise f would factor as $f' \circ \pi_n$ contradicting minimality of n. Now, calculate $C \cdot K_X$. If $\pi : \tilde{Y} \to Y$ is a blowup of a point p on a surface Y, and $\tilde{D}$ is an irreducible curve in $\tilde{Y}$ such that $\pi(\tilde{D})$ is a curve D, then we have $K_{\tilde{Y}} \cdot \tilde{D} = K_Y \cdot D + m \geq K_Y \cdot D$, where m is the multiplicity of D at p, i.e. $E \cdot \tilde{D}$. Equality holds iff $\tilde{D}$ doesn't intersect the exceptional divisor. Since f is composed of blowups, we get $K_X \cdot C \leq K_{\tilde{X}} \cdot E = -1$ with equality iff E doesn't meet any curve contracted by f. But in that case, f restricted to E is an isomorphism, so C is a rational curve with $K \cdot C = -1$, contradicting the minimality of X. So $K_X \cdot C \leq -2$, and $C^2 \geq 0$ by the genus formula. Now, this implies that all the plurigenera vanish, for if $|nK|$ contained an effective divisor D for $n \geq 1$, then $D \cdot C \geq 0$ by the useful lemma and $K_X \cdot C \geq 0$, a contradiction. If $q = 0$, Castelnuovo's theorem (for $q = 0, p_2 = 0$) implies that X is rational, excluded by hypothesis. If $q > 0$, $X \to \mathrm{Alb}\,(X)$ gives a surjective morphism $p : X \to B$ with connected fibers, where B is a smooth curve of genus $q > 0$. Since C is rational, C is contained in a fiber of p, and since $C^2 \geq 0$, we must have $F = rC$ for some r, so $C^2 = 0 \implies C \cdot K = -2$. Again, the genus formula gives $r = 1, g(F) = 0$ and C smooth, which by Noether-Enriques implies that X is ruled, with is also excluded. □

We now go on to separate surfaces into the following types.

(a) There is an integral curve C on X with $K \cdot C < 0$.
(b) For every integral curve C on K, we have $K \cdot C = 0$, i.e. $K \equiv 0$.
(c) $K^2 = 0, K \cdot C \geq 0$ for every integral curve C on X, and there is at least one integral curve C' s.t. $K \cdot C' > 0$.
(d) $K^2 > 0$, and $K \cdot C \geq 0$ for every integral curve C on X.

We will show that:

(1) X is in class (a) $\Leftrightarrow \kappa(X) = -\infty \Leftrightarrow p_4 = p_6 = 0 \Leftrightarrow p_{12} = 0$
(2) X is in class (b) $\Leftrightarrow \kappa(X) = 0 \Leftrightarrow 4K \sim 0$ or $6K \sim 0 \Leftrightarrow 12K \sim 0$.
(3) X is in class (c) $\Leftrightarrow \kappa(X) = 1 \Leftrightarrow |4K|$ or $|6K|$ has a strictly positive divisor at $K^2 \Leftrightarrow |12K|$ has a strictly positive divisor and $K^2 = 0$.
(4) X is in class (d) $\Leftrightarrow \kappa(X) = 2 \Leftrightarrow |2K| \neq \varnothing$.

Proof. We demonstrate this following Mumford, Mumford-Bombieri, and Badescu. First, let us see that every surface is exactly in one of the classes above. Mutual exclusivity is obvious. If X is not in any of the four classes, then $K^2 < 0$ and $K \cdot C \geq 0$ for every curve C on X. We can exclude this case as follows: let H be a hyperplane section, $D = aK + bH$ for a, b natural numbers. Then $D^2 = a^2K^2 + 2abK \cdot H + b^2H^2 = a^2P(b/a)$ for $P(t) = H^2t^2 + 2(K \cdot H)t + K^2$. By our hypothesis, P is an increasing function on $[0, \infty)$, is eventually positive, and $P(0) < 0$, implying that it has a unique root t_0. For $b/a > t_0$, $0 < a^2P(b/a) = D^2$. Also, for every integral C, $D \cdot C = a(K \cdot C) + b(H \cdot C) > 0$. By Nakai-Moishezon, such a D is ample, so nD is very ample for $n >> 0$ and $K \cdot D \geq 0$. Thus, for $t > t_0$, $(K \cdot H)t + K^2 \geq 0$, and by continuity, the same is true for $t = t_0$. $P(t_0) \geq H^2t_0^2 - K^2 > 0$, giving us a contradiction.

Now we begin to prove or equivalences. To show (i), we need to show that (a) $\implies$ X is ruled. In fact, we can replace (a) by saying that $\exists$ an effective divisor D on X s.t. $K \cdot D < 0$.

- Step 1: there is an ample H s.t. $K \cdot H < 0$. To see this, note that if $C^2 < 0$, then $K \cdot C + C^2 = 2p_a(C) - 2 \geq -2$, implying that $K \cdot C = C^2 = -1$ and so X is not minimal. Thus, $C^2 \geq 0$. Let H_1 be an ample divisor on X. Then, for all $n \geq 0$, $nC + H_1$ is ample by Nakai-Moishezon, and for $n >> 0$, $K \cdot (nC + H_1) < 0$ so we're done.
- Step 2: If $K^2 > 0$, then X is rational, hence ruled. Noether's formula gives $12\chi(OO_X) = K^2 + c_2 = K^2 + 2 - 2b_1 + b_2$. Since $p_g = 0$ (if $|nK|$ were effective, $nK \cdot H$ would be positive, contradicting Step 1), it follows that the Picard scheme is reduced, $b_1 = 2q$, and $10 = 8q + K^2 + b_2$. If $K^2 > 0$, then $q = 0$ or 1 is forced. If $q = 1$, then since $q = s = \dim \mathrm{Alb}\,(X)$, there is a morphism $X \to E$ to an elliptic curve, and so $b_2 \geq 2$ (Pic has the class of a fiber and class of a hyperplane section). This is impossible, so $q = 0$. By Castelnuovo, X is rational and thus ruled.
- Step 3: If $K^2 \leq 0$, then for all n, there is an effective divisor D on X s.t. $|D + K| = \varnothing$ and $\dim |D| \geq n$. To see this, Let H be an ample divisor s.t. $K \cdot H < 0$. For all n, $(nH + mK) \cdot H < 0$ for $m >> 0$ (depending on n), so $nH + mK$ can't be linearly equivalent to an effective divisor for $m >> 0$. Let m_n be a nonnegative integer s.t. $|nH + m_nK| \neq \varnothing$ but $|nH + (m_n + 1)K| = \varnothing$. Len $D_n \in |nH + m_nK|$, and write it as $D'_n + D''_n$, where each summand is positive and the components E of D'_n satisfy $E \cdot K < 0$, while those of D''_n satisfy $E \cdot K \geq 0$. Note that $E \cdot K < 0 \implies E^2 \geq 0$ (E not exceptional), so $(D'_n)^2 \geq 0$. Next, $|K - D'_n| \subset |K| = \varnothing$, so by Serre duality $H^2(\mathcal{O}_X(D'_n)) = 0$. Riemann-Roch gives that

$$\begin{aligned}
\dim |D'_n| &= h^0(\mathcal{O}_X(D'_n)) - 1 \geq \chi(\mathcal{O}_X(D'_n)) - 1 \\
&\geq \frac{(D'_n \cdot (D'_n - K))}{2} + \chi(\mathcal{O}_X) - 1 \\
&\geq \frac{-D'_n \cdot K}{2} + \chi(\mathcal{O}_X) - 1 \\
&\geq \frac{-D_n \cdot K}{2} + \chi(\mathcal{O}_X) - 1 \\
&\geq \frac{-n(H \cdot K)}{2} - \frac{m_nK^2}{2} + \chi(\mathcal{O}_X) - 1 \\
&\geq \frac{n}{2} + \chi(\mathcal{O}_X) - 1 \to \infty \text{ as } n \to \infty
\end{aligned} \tag{1}$$

Also, $|K + D'_n| \subset |K + D_n| = |nH + (m_n + 1)K| = \varnothing$.

- Step 4: If D is an effective divisor s.t. $|K+D| = \varnothing$, then the natural map $\operatorname{Pic}^0(X) \to \operatorname{Pic}^0(D)$ is surjective. To see this, note that $h^0(\mathcal{O}_X(K+D)) = h^2(\mathcal{O}_X(-D)) = 0$. Now, $0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$ gives that $H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_D)$ is surjective. These are the tangent spaces at 0 to the connected and reduced group schemes $\operatorname{Pic}^0(X)$ and $\operatorname{Pic}^0(D)$ ($\operatorname{Pic}^0(X)$ is reduced since $p_g = 0$ so $\Delta = 0$). Thus, the desired map is surjective.
- Step 5: If D is an effective divisor s.t. $|K+D| = \varnothing$ and if $D = \sum n_i E_i$, then
  1. All the E_i are nonsingular, and $\sum p_a(E_i) \leq q = h^1(X, \mathcal{O}_X)$.
  2. $\{E_i\}$ is a configuration of curves with no loops, and E_i intersect transversely.
  3. If $n_i \geq 2$¡ then either
     (a) E_i is rational,
     (b) $(E_i)^2 < 0$, or
     (c) E_i is an elliptic curve with $E_i^2 = 0$ and the normal bundle of E_i in X is nontrivial.

□