October 23, 2003

Cor. (of problem 6 on the homework - ps6): If X is an affine variety, $U \subset X$ an open set, then there is an affine variety Y andan open inclusion $U \hookrightarrow Y$ such that $\Gamma(Y, O_Y) \to \Gamma(U, O_U)$ is an isomorphism.

1 Back to dimension

We have $\dim X = \text{tr.deg } k(X)$ and equivalently, $\dim X$ is the length of the longest chain $\emptyset \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_r \subsetneq X$. Third equivalent definition: $\dim X$ is equal to the Krull dimension $O_{X,x}$ for any point $x \in X$.

Last time we showed that if X is a variety, $0 \neq f \in \Gamma(X, O_X)$ then the dimension of an irreducible component of $V(f)$ is dimension $\dim X - 1$.

Question: Given an irreducible $Z \subset X$ of codimension r (dimension $\dim X - r$) is there $f_1, \ldots, f_r \in \Gamma(X, O_X)$ such that $Z = V(f_1, \ldots, f_r)$? As it turns out, not always. First of all, it definitely doesn't work when $X = \mathbb{P}^n$ since we proved the only functions in $\Gamma(\mathbb{P}^n, O_{\mathbb{P}^n})$ were constant.

Example. Let $C \subset \mathbb{P}^3$ be the twisted cubic. Then $\dim C = 1$ and C is a component of $H_1 \cap H_2$ where $H_i \subset \mathbb{P}^3$ are hypersurfaces, $V(xz - y^2) \cap V(yw - z^2) = C \cup L$. But there are no two hypersurfaces we can pick that give C exactly.

Cor. Let $X \subset Z$ which is an irreducible component of $V(f_1, \ldots, f_r), f_i \in \Gamma(X, O_X)$ not zero. Then the codimension of Z is at most r.

Pf. Induction. Let $Z' \subset V(f_1, \ldots, f_r)$ be the irreducible component containing Z, et cetera.

Cor. Let X be affine, $Z \subset X$ irreducible, closed, of codimension r. Let $R = \Gamma(X, O_X)$. Then there exist $f_1, \ldots, f_r$ such that Z is an irreducible component of $V(f_1, \ldots, f_r)$.

Pf. THere is a chain $Z = Z_r \subsetneq Z_{r-1} \subsetneq \ldots \subsetneq Z_0 = X$ such that Z_i is irreducible and closed of codimension i. Induction.

2 Krull Dimension

Def. Recall: A local ring, $m \subset A$ a maximal ideal. Then the *Krull dimension* of A is the maximum length of a chain of prime ideals $P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_n = m$.

Let $Z \subset X$ be irreducible, closed. We want to relate the codimension of Z to the Krull dimension.

If you remember all this stuff then the rest of the lecture is pretty trivial. The codimension of Z is equal to the Krull dimension of $O_{X,Z} = \Gamma(X, O_X)_{I(Z)}$.

Prop. Let X be affine, $R = \Gamma(X, O_X)$, a UFD. Then any closed $Z \subset X$ of codimension 1 is $V(f)$ for some $f \in R$.

Pf. Let $P \subset R$ be minimal, which implies that $P = (f_P)$ for some $f_P \in R$ (since R is a UFD this is the case). So say $Z = Z_1 \cup \ldots \cup Z_r$ where Z_i irreducible. Then $I(Z_i) = (f_i)$, and $Z = V(\prod_i f_i)$.

2.1 Projective versions.

Let $X \subset \mathbb{P}^n$ be a projective variety, $I(X) \subset k[X_0, \ldots, X_n]$ where $\dim X > 0$.

Theorem. Let $f \in k[\underline{X}]$ homogeneous, not constant, then $X \cap V(f) \neq \emptyset$ and has pure codimension 1.

Define $X^* \subset \mathbb{A}^{n+1}$ the cone of X defined by $I(X)$. On our next homework we will show $\dim X^* = \dim X + 1 \geq 2$. Look at $V(f)^* \subset \mathbb{A}^{n+1}$. Look at $V(f)^* \cap X^*$. We know this is nonempty (it contains $(0, \ldots, 0)$ for instance), so this has pure dimension at least 1, so it must contain at least some other point. Thus, $X \cap V(f) \neq \emptyset$.