October 2, 2003

1 Homework Review

Problem 1. The point was to play with / understand the associated sheaf. If we have $f : F \to G$ what should the image be? The image should certainly be a *subsheaf* of G, but it should be the smallest such that $f : F \to I$ makes sense. How do we find this? For all U we have a map $F(U) \to G(U)$. If we let $I(U)$ be this, we only get a presheaf; to fix this we take the associated sheaf. This is the right definition of image since $I \subset H \subset G$ for any subsheaf H and furthermore, $I^a \subset H$.

Problem 2. Not really to be used later. Exercise in limits.

Problem 8: I understood this one, actually.

Problem 4: $X \xrightarrow{f} Y$ a morphism corresponds to a map of rings $\Gamma(Y) \xrightarrow{\rho} \Gamma(X)$ finite. A point y corresponds to a maximal ideal $m_y \subset \Gamma(Y, O_Y)$. $f^{-1}(X)$ corresponds to some set of maximal ideals m in $\Gamma(X, O_X)$ with $\rho^{-1}(m) = m_y$. This set is in bijection with the set of maximal ideals in $R = \Gamma(X, O_X)/\rho(m_y)\Gamma(X, O_X)$ (not hard). This ring R is strange: it is a k-algebra, but is actually a finite-dimensional vector space. This makes it Artinian and Artinian rings have only finitely man maximal ideals. Thus our set can only be finite.

Problem 7. Why is $X = \mathbb{A}^2 - \{(0,0)\}$ not an affine variety? There is a natural map $\Gamma(X, O_{\mathbb{A}^2}) \leftarrow \Gamma(\mathbb{A}^2 O_{\mathbb{A}^2})$. If X is an affine variety then $X \hookrightarrow \mathbb{A}^2$ is an isomorphism.

2 Back to varieties

Def. A pair (X, O_X) is a prevariety where X is a topological space and O_X is a sheaf of k-valued functions if:

1. X is connected, and
2. There is a finite covering $X = \cup_{i=1}^n U_i$ where for each i, $(U_i, O_X|_{U_i})$ is a variety.

Def. An open set $U \subset X$ such that $(U, O_X|_U)$ is an affine variety is called an *affine open* or *open affine* (set). So another definition of a prevariety is a pair (X, O_X) where X is connected and has a finite open affine cover.

Ex. $(\mathbb{A}^1, O_{\mathbb{A}^1}) \supset (\mathbb{A}^1 \setminus 0, O_{\mathbb{A}^1}|_{\mathbb{A}^1 \setminus 0})$, and we want to put two of these together into $\hookrightarrow (\mathbb{A}^1, O_{\mathbb{A}^1})$. In general, we can glue varieties together as follows. If we have $U \hookrightarrow X$ and $V \hookrightarrow Y$ where $U \cong V$, then we can make a variety $(X \cup Y)/(U - V)$. For this gluing, we get the line except with the point at the origin duplicated: $X = \mathbb{A}^1 \cup \{0'\}$ where the natural open covering is $U_1 = \mathbb{A}^1$ and $U_2 = \mathbb{A}^1 - \{0\} \cup \{0'\}$. This corresponds to a function which has two limits at a point ("non-separated" functions). This is a prevariety.

The gluing is defined by $U = V = \mathbb{A}^1 - \{0\}$, and $X = Y = \mathbb{A}^1$.

Ex. Another way to do this gluing is to use the isomorphism $U \cong V$ defined by $\lambda \mapsto \lambda^{-1}$. Now when we glue X and Y we get $\mathbb{P}^1$! If you think about it, this ends up being $\mathbb{A}^2/(\lambda, 1) \sim (1, \lambda^{-1})$, which is just the projective line. It is probably easier to visualize this if we start with $\mathbb{P}^1$ and look at $X = U_1$ and $Y = U_2$ as copies of $\mathbb{A}^1$ and how they agree on the overlap.

Def. A top. space X is *irreducible* if it is not a union of two proper closed subsets. Affine varieties (remember, they are irreducible affine alg. sets) are irreducible top. spaces.

Lemma. If (X, O_X) is a prevariety, then X is irreducible. (Not obvious!)

Proof. This is equivalent to saying that any two open sets intersect: if this is true, then $C_1 \cup C_2 = \overline{U_1 \cap U_2} \neq X$.

Let $V \subset X$ open, $V \neq \emptyset$. Let $U_1 = \cup_{W \cap V \neq \emptyset} W, U_2 = \cup_{W \cap V = \emptyset} W$, where we take unions over only such open sets. Now X is connected so $U_1 \cap U_2 \neq \emptyset$, so let $y \in U_1 \cap U_2$.

There exist affine opens W_1, W_2 containing y such that $W_1 \cap V$ and $W_2 \cap V$ are each nonempty. ...

Basically, we reduce to the affine case; if there are two closed sets that cover X, we can intersect them in some good way with W_1 and W_2 where W_1, W_2 are affine open sets, and then the irreducibility of W_1 and W_2 prove it.

Cor. (i) Every open set is dense, (ii) any two open sets intersect. We prove (i) by noting that $X = \overline{U} \cup U^C$ where $\overline{U}$ is the closure; if U is a proper open set then U^C is a proper closed set so $\overline{U}$ must not be proper, hence it is the whole space X.

Lemma. Let X be a prevariety. Then closed sets satisfy the DCC. That is, any sequence $X \supset Z_1 \supset \ldots \supset Z_n \ldots$ then it stabilizes eventually.

Proof. We know $X = U_1 \cup \ldots \cup U_n$ and for each U_i we know $Z_1 \cap U_i \supset \ldots$ is eventually stable, so if we take the maximum such n this stabilizes the sequence in X.

Lemma. Any open set $U \subset X$ is quasi-compact. Obviously: you just intersect the open cover of U with the affine cover, then it's true.

Def. The *function field* $k(X)$ of a prevariety X is $\overset{\lim}{\rightarrow}_{U \subset X} O_X(U)$. [Is this the intersection of all the stalks? No, it's something else.] Elements in $k(X)$ are called *rational functions.*

Lemma. If $U \subset X$ is an affine open set, then $k(X) = \text{Frac}\ (\Gamma(U, O_U))$. Proof: we already proved $\text{Frac}\ (\Gamma(U, O_U)) = \overset{\lim}{\rightarrow}_{V \subset U} O_U(U)$.

Lemma. Let X be a prevariety. If $U \subset X$ is open, then $(U, O_X|_U)$ is also a prevariety.

Proof. The first thing is to prove that it is connected; if it is not then those would be two open sets in X that do not meet. Next, choose $\cup_{i=1}^n U_i = X$ such that each U_i is an affine variety. It is enough to show that $U \cap U_i$ is a prevariety.