Proof of the spectral theorem

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1 Spectral theorem

Here is the definition of selfadjoint, more or less exactly as in the text.

Definition 1.1. Suppose V is a (real or complex) inner product space. A linear transformation $S \in \mathcal{L}(V)$ is *selfadjoint* if

$$\langle Sv, w \rangle = \langle v, Sw \rangle \qquad (v, w \in V).$$

The point of these notes is to explain a proof (somewhat different from the one in the book) of

Theorem 1.2 (Spectral theorem)**.** *Suppose V is a finite-dimensional real or complex vector space. The linear operator $S \in \mathcal{L}(V)$ is selfadjoint if and only if V is the orthogonal direct sum of the eigenspaces of S for real eigenvalues:*

$$V = \sum_{\lambda \in \mathbb{R}} V_\lambda.$$

Here by definition

$$V_\lambda = \{v \in V \mid Sv = \lambda v\}$$

is the eigenspace for the eigenvalue λ. The orthogonality requirement means

$$\langle v, w \rangle = 0 \qquad (v \in V_\lambda, w \in V_\mu, \lambda \neq \mu).$$

The theorem says first of all that a selfadjoint operator is diagonalizable, and that all the eigenvalues are real.

The *orthogonality* of the eigenspaces is important as well. Orthogonal decompositions are easy to compute with, and they are in a certain sense very "rigid" and stable. If L_1 and L_2 are distinct lines in $\mathbb{R}^2$, then automatically $\mathbb{R}^2 = L_1 \oplus L_2$. That is, any vector $v \in \mathbb{R}^2$ can be written uniquely as

$$v = \ell_1 + \ell_2, \qquad (\ell_i \in L_i). \tag{1.3}$$

If the two lines point in nearly the same direction, then writing a small vector v may require large vectors in L_1 and L_2. For example, if

$$L_1 = x \text{ axis} = \mathbb{R} \cdot (1,0), \qquad L_2 = \mathbb{R} \cdot (1, 1/10), \tag{1.4}$$

then

$$(0, 1/10) = (-1, 0) + (1, 1/10).$$

In a computational world, this means that small errors in v may correspond to large errors in the "coordinates" ℓ_i. If on the other hand the lines L_i are orthogonal, then

$$\|v\|^2 = \|\ell_1\|^2 + \|\ell_2\|^2, \tag{1.5}$$

so that small v correspond to small ℓ_i.

The proof that the conditions in the theorem imply selfadjointness is straightforward, and I won't do it. What's interesting and difficult is the proof that selfadjointness implies the eigenspace decomposition. This is based on the next two lemmas.

Lemma 1.6 (Orthogonal invariants)**.** *Suppose S is a selfadjoint operator on an inner product space V, and $U \subset V$ is an S-invariant subspace:*

$$Su \in U, \qquad (u \in U).$$

Then the orthogonal complement $U^\perp$ is also S-invariant.

Lemma 1.7 (Maximal eigenspaces)**.** *Suppose S is a selfadjoint operator on a finite-dimensional inner product space V. The function*

$$s(v) = \langle Sv, v \rangle$$

takes real values on on V. If v_0 is a maximum for s on the unit sphere

$$\textit{sphere} = \{v \in V \mid \langle v, v \rangle = 1\},$$

then v_0 is an eigenvector of S with (real) eigenvalue $s(v_0)$.

We postpone the proofs of these two lemmas for a moment. Assuming them...

Proof of Spectral Theorem. Recall that we are proving only that a selfadjoint operator has the orthogonal eigenspace decomposition described. We proceed by induction on $\dim V$. If $\dim V = 0$, then $S = 0$ and there are no eigenvalues; the theorem says that the zero vector space is an empty direct sum, which is true by definition.

So suppose $\dim V > 0$, and that the theorem is known for spaces of lower dimension. Because $V \neq 0$, there are nonzero vectors, so there are vectors of length one, and the unit sphere in Lemma 1.7 is not empty. The function $s(v)$ is continuous (if we choose a basis for V, s is a quadratic polynomial function of the coordinates). Any continuous function must have a maximum on a closed bounded set like the unit sphere, so s has a maximum at some point v_0. (This is a hard fact, proved in 18.100; you're not necessarily supposed to know it to understand this course. But the Spectral Theorem is a hard theorem, so you need to do something difficult somewhere. The proof in the text uses the existence of eigenvalues on complex vector spaces, which amounts to the Fundamental Theorem of Algebra. That's hard too.)

According to Lemma 1.7,

$$v_0 \in V_{s(v_0)}; \tag{1.8}$$

so the eigenspace is not zero. Because an eigenspace is obviously an invariant subspace, Lemma 1.6 implies that

$$V = V_{s(v_0)} \oplus V_{s(v_0)}^{\perp}, \tag{1.9}$$

an orthogonal direct sum of S-invariant subspaces. Because $V_{s(v_0)}$ is not zero, it has positive dimension, so

$$\dim V_{s(v_0)}^{\perp} = \dim V - \dim V_{s(v_0)} < \dim V. \tag{1.10}$$

By inductive hypothesis, $V_{s(v_0)}^{\perp}$ is an orthogonal direct sum of eigenspaces of S. Inserting this orthogonal decomposition in (1.9) gives the complete orthogonal decomposition into eigenspaces that we want. $\square$

The proof of Lemma 1.6 is extremely easy: you just write down what the lemma says is true, and see (using the definition of selfadjoint) that it's obviously true.

Proof of Lemma 1.7. Assume that v_0 is a maximum for s on the unit sphere. We are going to show that

$$\text{if } \langle v_0, u\rangle = 0, \text{ then } \operatorname{Re}\langle S(v_0), u\rangle = 0. \tag{1.11}$$

If we can prove this, then (applying it in the complex case to all the vectors $e^{i\theta}u$) we will know also that

$$\text{if } \langle v_0, u\rangle = 0, \text{ then } \langle S(v_0), u\rangle = 0.$$

This says that everything orthogonal to the line $F \cdot v_0$ is also orthogonal to $S(v_0)$: that is

$$u \in (F \cdot v_0)^\perp \implies \langle u, S(v_0)\rangle = 0.$$

That is,

$$S(v_0) \in \left((F \cdot v_0)^\perp\right)^\perp = F \cdot v_0.$$

This last equation is precisely the statement that v_0 is an eigenvector of S:

$$S(v_0) = \lambda v_0 \qquad (\text{some } \lambda \in F).$$

The eigenvalue λ is

$$\frac{\langle S(v_0), v_0\rangle}{\langle v_0, v_0\rangle} = \langle S(v_0), v_0\rangle = s(v_0),$$

as we wished to show.

It remains to prove (1.11). This condition is true for u if and only if it is true for any multiple of u, and it is true for $u = 0$; so it is enough to prove it for vectors u of length 1. We want to use the fact that v_0 is a maximum for s on the unit sphere, so we need to cook up more vectors on the unit sphere. Because u and v_0 are orthogonal unit vectors, the Pythagorean Theorem says that

$$v_t = \cos(t)v_0 + \sin(t)u \qquad (t \in \mathbb{R}) \tag{1.12}$$

are all unit vectors. The notation is chosen so that the new definition of v_0 is equal to the old definition of v_0; so the function of one real variable

$$f(t) = s(v_t) = \langle Sv_t, v_t\rangle \tag{1.13}$$

has a maximum at $t = 0$. We're going to use calculus on the function f. The formula is

$$\begin{aligned} f(t) &= \langle Sv_t, v_t\rangle \\ &= \cos^2(t)\langle Sv_0, v_0\rangle + \sin^2(t)\langle Su, u\rangle + \cos(t)\sin(t)\left[\langle Sv_0, u\rangle + \langle Su, v_0\rangle\right]. \end{aligned}$$

The selfadjointness of S says that

$$\langle Su, v_0\rangle = \overline{\langle Sv_0, u\rangle},$$

so we get

$$f(t) = \cos^2(t)\langle Sv_0, v_0\rangle + \sin^2(t)\langle Su, u\rangle + 2\cos(t)\sin(t)\left[\mathrm{Re}\langle Sv_0, u\rangle\right].$$

Consequently f is differentiable, and

$$f'(0) = 2[\mathrm{Re}\langle Sv_0, u\rangle].$$

Since 0 is a maximum for the differentiable function f, the derivative must be zero at 0; and this is (1.11). $\square$