18.600: Lecture 6

Conditional probability

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Outline

Definition: probability of A given B

Examples

Multiplication rule

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- Definition makes sense even without "equally likely" assumption.

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- Probability plane will come eventually, given plane not here yet.

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- Study participants believe blue taxi at fault, say witness correct with 80 percent probability.

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- Another example: roll die and let E_i be event that the roll *does not lie* in $\{1, 2, \dots, i\}$. Then $P(E_i) = (6 - i)/6$ for $i \in \{1, 2, \dots, 6\}$.

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- What is $P(E_4|E_1 E_2 E_3)$ in this case?

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- We have $P\big((1, 2)\big) = P\big((1, 3)\big) = 1/6$ and $P\big((2, 3)\big) = P\big((3, 2)\big) = 1/3$. Given host points to door 2, probability prize behind 3 is $2/3$.

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- Easy to see answer is $13/27$.