18.600 Midterm 1, Spring 2017 Solutions

1. (20 points) Let X be the number on a standard die roll (assuming values in $\{1,2,3,4,5,6\}$ with equal probability). Let Y and Z be the numbers on two independent rolls of the same die (so X, Y, and Z are independent of each other). Compute the following:

(a) $E[X]$ and $E[X^2]$ **ANSWER:** $E[X] = \frac{1}{6}(1+2+3+4+5+6) = 7/2$ and $E[X^2] = \frac{1}{6}(1+4+9+16+25+36) = 91/6$

(b) $\mathrm{Var}[X]$ **ANSWER:** $E[X^2] - E[X]^2 = 91/6 - 49/4 = 35/12$

(c) $E[X^2 + 2Y^2 + 3Z^2]$ **ANSWER:** $E[X^2] + 2E[Y^2] + 3[Z^2]$ (by linearity of expectation) which is $6E[X^2] = 91$

(d) The conditional probablity $P[X = 1|X = Y]$ **ANSWER:** $P[X = 1, X = Y]/P[X = Y] = \frac{1/36}{1/6} = 1/6$

2. (10 points) An urn contains 5 black balls and 5 white balls and 5 red balls and 5 yellow balls. If a collection of 8 balls is chosen uniformly at random from the urn, what is the probability that the collection contains exactly 2 balls of each color? **ANSWER:** $\binom{5}{2}^4/\binom{20}{8}$

3. (20 points) Let M be the event that a person has a certain medical condition and let T be the event that a test of that condition comes back positive. Assume that the person has a 1/20 chance to have the condition and given that the person has the condition, there is a 90 percent chance the test will identify the condition correctly. Assume also that there is 20 percent chance the test will come back positive given that the person doesn't have the condition. In symbols, we can write the above assumptions as $P(M) = 1/20$ and $P(T|M) = 9/10$ and $P(T|M^c) = 1/5$. Now compute the following:

(a) $P(T)$ (i.e., overall likelihood test is positive) **ANSWER:** $P(M)P(T|M) + P(M^c)P(T|M^c) = 9/200 + 19/100 = 47/200$

(b) $P(M|T)$ (i.e., likelihood person has condition given test is positive) $P(MT)/P(T) = P(M)P(T|M)/P(T) = (9/200)/(47/200) = 9/47$

(c) $P(M|T^c)$ (i.e., likelihood person has condition given test is negative) $P(MT^c)/P(T^c) = P(M)P(T^c|M)/P(T^c) = (1/200)/(153/200) = 1/153$

4. (10 points) A standard deck of cards has 4 suits, and 13 different cards of each suit (52 total). We know that there are $\binom{52}{13}$ ways to choose an (unordered) set of 13 cards from this deck. How many ways are there to choose an (unordered) set of 13 cards with the property that 7 of the cards belong to the same suit (with the other 6 *not* belonging to that suit)? **ANSWER:** Choose the special suit (4 choices) then choose seven from that suit ($\binom{13}{7}$ choices), then choose 6 from the 39 not of that suit ($\binom{39}{6}$ choices). Overall number is $4\binom{13}{7}\binom{39}{6}$

5. (10 points) You have 12 pieces of pizza and 4 people at a party. How many ways are there to divide the 12 pieces among the four people? In other words, how many ordered four-tuples of non-negative integers (a_1, a_2, a_3, a_4) satisfy $a_1 + a_2 + a_3 + a_4 = 12$? **ANSWER:** Stars and bars argument gives $\binom{15}{3}$

6. (20 points) Harry is a (relatively untalented) basketball player practicing free throws. Each time Harry attempts a shot, Harry has a 1/100 probability of making the shot (independently of all other shots taken). Let X be the number of shots that Harry makes after 100 attempts.

(a) Compute the variance $\mathrm{Var}[X]$. (Given an exact answer, not an approximation.) **ANSWER:** $npq = 100(1/100)(99/100) = 99/100$

(b) Use a Poisson approximation to give an estimate for the probability that $X = 2$. **ANSWER:** $e^{-\lambda}\lambda^k/k!$ with $\lambda = 1$ and $k = 2$ gives $\frac{1}{2e}$

(c) Harry is hoping the event $X > 1$ will occur, since this would mean that (thanks to good luck) Harry made *more* shots than he expected to make (which might impress any talent scouts who happened to present). Use a Poisson approximation to estimate $P\{X > 1\}$. **ANSWER:** Using same Poisson approximation with $k = 1$ and $k = 0$ we find $P\{X = 0\} \approx P\{X = 1\} \approx 1/e$. So $P\{X > 1\} = 1 - P\{X = 0\} - P\{X = 1\} \approx 1 - 2/e$

(d) Using the same Poisson approximation, estimate $P\{X < 1\}$. Is this larger or smaller than (or the same as) your estimate for $P\{X > 1\}$? **ANSWER:** As computed above $P\{X < 1\} = P\{X = 0\} \approx 1/e$. Since $e < 3$ we have $1 - 2/e < 1/e$. This means that Harry is more likely to be unlucky (and score less than expected) than to be lucky (and score more than expected), which sounds bad for Harry. But on the plus side, Harry can at worst score *one* less than expected, but Harry can at best score *a lot* more than expected.

7. (10 points) Fix an integer $n \geq 3$. Let σ be a random permutation chosen uniformly from the set of $n!$ permutations of $\{1, 2, \ldots, n\}$. Let $X = X(\sigma)$ be the number of cycles of length 3 in the permutation σ. Compute $E[X]$.

Note: Here is some notation you can use in your answer if it is helpful. Given integers $i < j < k$, let $E_{i,j,k}$ be the event that the integers i, j, k are part of a length three cycle (which means that *EITHER* $\sigma(i) = j$, $\sigma(j) = k$, and $\sigma(k) = i$ *OR* $\sigma(i) = k$, $\sigma(k) = j$, and $\sigma(j) = i$). Let $X_{i,j,k}$ be the indicator random variable $1_{E_{i,j,k}}$.

ANSWER: $E[X_{i,j,k}] = 2\frac{1}{n}\frac{1}{n-1}\frac{1}{n-2}$ for each $1 \leq i < j < k \leq n$. We have $E[X] = \sum E[X_{i,j,k}]$ where the sum ranges over $\binom{n}{3}$ such terms, so we end up with $2\frac{1}{n}\frac{1}{n-1}\frac{1}{n-2}\binom{n}{3} = 2\frac{n(n-1)(n-2)}{n(n-1)(n-2)3!} = 1/3$