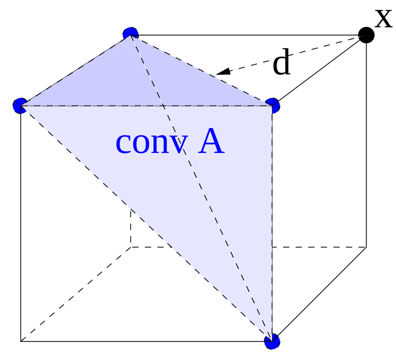
Let $\mathcal{X} = \{0,1\}$, $(x_1,\ldots,x_n) \in \{0,1\}^n$, $\mathbb{P}(x_i = 1) = p$, and $\mathbb{P}(x_i = 0) = 1-p$. Suppose $A \subseteq \{0,1\}^n$. What is $d(A,x)$ in this case?



For a given x, take all $y \in A$ and compute s:

$$\begin{array}{rcccc} x = (& x_1, & x_2, & \ldots, & x_n) \\ & = & \neq & \ldots & = \\ y = (& y_1, & y_2, & \ldots, & y_n) \\ s = (& 0, & 1, & \ldots, & 0) \end{array}$$

Build conv $V(A,x) = U(A,x)$. Finally, $d(A,x) = \min\{|x-u|^2; u \in \text{conv } A\}$

Theorem 34.1. *Consider a convex and Lipschitz $f : \mathbb{R}^n \mapsto \mathbb{R}$, $|f(x) - f(y)| \le L|x-y|$, $\forall x,y \in \mathbb{R}^n$. Then*

$$\mathbb{P}\left(f(x_1,\ldots,x_n) \ge M + L\sqrt{t}\right) \le 2e^{-t/4}$$

and

$$\mathbb{P}\left(f(x_1,\ldots,x_n) \le M - L\sqrt{t}\right) \le 2e^{-t/4}$$

where M is median of f: $\mathbb{P}(f \ge M) \ge 1/2$ and $\mathbb{P}(f \le M) \ge 1/2$.

Proof. Fix $a \in \mathbb{R}$ and consider $A = \{(x_1,\ldots,x_n) \in \{0,1\}^n,\ f(x_1,\ldots,x_n) \le a\}$. We proved that

$$\mathbb{P}\left(\underbrace{d(A,x) \ge t}_{\text{event } E}\right) \le \frac{1}{\mathbb{P}(A)} e^{-t/4} = \frac{1}{\mathbb{P}(f \le a)} e^{-t/4}$$

$$d(A,x) = \min\{|x-u|^2; u \in \text{conv } A\} = |x-u_0|^2$$

for some $u_0 \in \text{conv } A$. Note that $|f(x) - f(u_0)| \le L|x-u_0|$.

Now, assume that x is such that $d(A,x) \le t$, i.e. complement of event E. Then $|x-u_0| = \sqrt{d(A,x)} \le \sqrt{t}$. Hence,

$$|f(x) - f(u_0)| \le L|x-u_0| \le L\sqrt{t}.$$

So, $f(x) \leq f(u_0) + L\sqrt{t}$. What is $f(u_0)$? We know that $u_0 \in \text{conv } A$, so $u_0 = \sum \lambda_i a_i$, $a_i \in A$, and $\lambda_i \geq 0$, $\sum \lambda_i = 1$. Since f is convex,

$$f(u_0) = f\left(\sum \lambda_i a_i\right) \leq \sum \lambda_i f(a_i) \leq \sum \lambda_i a = a.$$

This implies $f(x) \leq a + L\sqrt{t}$. We proved

$$\{d(A,x) \leq t\} \subseteq \{f(x) \leq a + L\sqrt{t}\}.$$

Hence,

$$1 - \frac{1}{\mathbb{P}(f \geq a)} e^{-t/4} \leq \mathbb{P}(d(A,x) \leq t) \leq \mathbb{P}\left(f(x) \leq a + L\sqrt{t}\right).$$

Therefore,

$$\mathbb{P}\left(f(x) \geq a + L\sqrt{t}\right) \leq \frac{1}{\mathbb{P}(f \geq a)} e^{-t/4}.$$

To prove the first inequality take $a = M$. Since $\mathbb{P}(f \leq M) \geq 1/2$,

$$\mathbb{P}\left(f(x) \geq M + L\sqrt{t}\right) \leq 2e^{-t/4}.$$

To prove the second inequality, take $a = M - L\sqrt{t}$. Then

$$\mathbb{P}(f \geq M) \leq \frac{1}{\mathbb{P}\left(f \leq M - L\sqrt{t}\right)} e^{-t/4},$$

which means

$$\mathbb{P}\left(f(x) \leq M - L\sqrt{t}\right) \leq 2e^{-t/4}.$$

Example 34.1. Let $H \subseteq \mathbb{R}^n$ be a bounded set. Let

$$f(x_1, \ldots, x_n) = \sup_{h \in \mathcal{H}} \left| \sum_{i=1}^n h_i x_i \right|.$$

Let's check:

(1) convexity:

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= \sup_{h \in \mathcal{H}} \left| \sum_{i=1}^n h_i(\lambda x_i + (1-\lambda) y_i) \right| \\ &= \sup_{h \in \mathcal{H}} \left| \lambda \sum_{i=1}^n h_i x_i + (1-\lambda) \sum_{i=1}^n h_i y_i \right| \\ &\leq \lambda \sup_{h \in \mathcal{H}} \left| \sum_{i=1}^n h_i x_i \right| + (1-\lambda) \sup_{h \in \mathcal{H}} \left| \sum_{i=1}^n h_i y_i \right| \\ &= \lambda f(x) + (1-\lambda) f(y) \end{aligned}$$

(2) Lipschitz:

$$
\begin{aligned}
|f(x)-f(y)| &= \left| \sup_{h\in\mathcal{H}} \left|\sum_{i=1}^n h_i x_i\right| - \sup_{h\in\mathcal{H}} \left|\sum_{i=1}^n h_i y_i\right| \right| \\
&\le \sup_{h\in\mathcal{H}} \left|\sum_{i=1}^n h_i (x_i - y_i)\right| \\
&\le \text{(by Cauchy-Schwartz)} \quad \sup_{h\in\mathcal{H}} \sqrt{\sum h_i^2}\sqrt{\sum (x_i-y_i)^2} \\
&= |x-y| \qquad \underbrace{\sup_{h\in\mathcal{H}} \sqrt{\sum h_i^2}}_{L=\text{Lipschitz constant}}
\end{aligned}
$$

We proved the following

Theorem 34.2. *If M is the median of $f(x_1,\ldots,x_n)$, and $x_1,\ldots,x_n$ are i.i.d with $\mathbb{P}(x_i=1)=p$ and $\mathbb{P}(x_i=0)=1-p$, then*

$$\mathbb{P}\left(\sup_{h\in\mathcal{H}}\left|\sum_{i=1}^n h_i x_i\right| \ge M + \sup_{h\in\mathcal{H}}\sqrt{\sum h_i^2}\cdot\sqrt{t}\right) \le 2e^{-t/4}$$

and

$$\mathbb{P}\left(\sup_{h\in\mathcal{H}}\left|\sum_{i=1}^n h_i x_i\right| \le M - \sup_{h\in\mathcal{H}}\sqrt{\sum h_i^2}\cdot\sqrt{t}\right) \le 2e^{-t/4}$$

□