18.435/2.111 Homework # 1 Solutions

Solution to 2.59: If we have a qubit in the state $|\,0\rangle$, and we measure the observable σ_x, we project onto the two eigenvectors of σ_x, which are $|\,+\rangle = \frac{1}{\sqrt{2}}(|\,0\rangle + |\,1\rangle)$, with eigenvalue 1, and $|\,-\rangle = \frac{1}{\sqrt{2}}(|\,0\rangle - |\,1\rangle)$, with eigenvalue -1. It is easy to check that $|\,0\rangle = \frac{1}{\sqrt{2}}(|\,+\rangle + |\,-\rangle)$, so the probability of seeing $|\,+\rangle$ and $|\,-\rangle$ is each $\frac{1}{2}$. We thus observe $+1$ with probability $\frac{1}{2}$ and -1 with probability $\frac{1}{2}$. The expectation is 0 and the standard deviation is 1.

Solution to 2.60: We have

$$\vec{v}\cdot\vec{\sigma} = v_x\sigma_x + v_y\sigma_y + v_z\sigma_z.$$

Consider

$$(\vec{v}\cdot\vec{\sigma})^2$$

By using the relations $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I$, and the fact that any two distinct Pauli matrices anticommute, i.e., $\sigma_x\sigma_y = -\sigma_y\sigma_x$, we can see that

$$(\vec{v}\cdot\vec{\sigma})^2 = (v_x^2 + v_y^2 + v_z^2)I = I.$$

Therefore, its eigenvalues must be ± 1. It is easy to check that $\vec{v}\cdot\vec{\sigma}$ is not $\pm I$, so one eigenvalue must be $+1$ and the other must be -1. Now, if we let the eigenvectors be $|\,\phi_+\rangle$ and $|\,\phi_-\rangle$, we have that

$$\vec{v}\cdot\vec{\sigma} = |\,\phi_+\rangle\,\langle\phi_+\,| - |\,\phi_-\rangle\,\langle\phi_-\,|$$

and so

$$(I + \vec{v}\cdot\vec{\sigma})/2 = |\,\phi_+\rangle\,\langle\phi_+\,| = P_+,$$

and similarly for the -1 eigenvector.

Solution to 2.61: The expectation of the observable $\vec{v}\cdot\vec{\sigma}$ when $|\,0\rangle$ is measured is

$$\langle 0\,|\,\vec{v}\cdot\vec{\sigma}\,|\,0\rangle \;\;=\;\; v_x\,\langle 0\,|\,\sigma_x\,|\,0\rangle + v_y\,\langle 0\,|\,\sigma_y\,|\,0\rangle + v_z\,\langle 0\,|\,\sigma_z\,|\,0\rangle\,.$$

This is easily seen to be equal to v_z. Thus,

$$v_z = \text{Prob}(+1) - \text{Prob}(-1)$$

and using the fact that the two probabilities add to 1 gives

$$\text{Prob}(+1) = (v_z + 1)/2.$$

Problems 1–3 deal with what is known as the GHZ state (after Greenberger, Horne and Zeilinger), and the proof of non-locality using this state which was discovered by Greenberger, Horne, Shimony and Zeilinger.

Solution to 1: We wish to measure the GHZ state in the $|+\rangle, |-\rangle$ basis, where $|+\rangle$ and $|-\rangle$ are defined as in the solution to problem 2.59 above. Using the distributive law and the equations

$$\begin{aligned} |0\rangle &= \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \\ |0\rangle &= \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle), \end{aligned}$$

we can perform a change of basis to find that

$$\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) = \frac{1}{2}(|+++\rangle + |--+\rangle + |-+-\rangle + |+--\rangle).$$

Thus, the probability of seeing each of $|+++\rangle$, $|--+\rangle$, $|-+-\rangle$ and $|+--\rangle$ is $\frac{1}{4}$, and the probability of seeing any of the other states (those with an odd number of $-$'s) is 0. This gives us that the expected value of the observable $\sigma_x(1) \otimes \sigma_x(2) \otimes \sigma_x(3)$ is $+1$, since $|+\rangle$ and $|-\rangle$ are the eigenvectors of σ_x, and this observable is $+1$ on all four states above. This can also be seen by explicitly taking the tensor product of the Pauli matrices σ_x, which is

$$\sigma_x(1) \otimes \sigma_x(2) \otimes \sigma_x(3) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and applying it to the GHZ state, which is $\frac{1}{\sqrt{2}}(1,0,0,0,0,0,0,1)$. We obtain

$$\langle \mathrm{GHZ} | \sigma_x(1) \otimes \sigma_x(2) \otimes \sigma_x(3) | \mathrm{GHZ} \rangle = 1.$$

Since its eigenvalues are ± 1, this means that the observable $\sigma_x(1) \otimes \sigma_x(2) \otimes \sigma_x(3)$ is 1 with probability 1.

If we measure the first qubit in the $|+\rangle, |-\rangle$ basis and the second and third in the $|+I\rangle, |-I\rangle$ basis, where $|\pm I\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle)$, we can again use the distributive law to make an explicit change of basis. We now have

$$\begin{aligned} |0\rangle &= \frac{1}{\sqrt{2}}(|+I\rangle + |-I\rangle), \\ |1\rangle &= \frac{-i}{\sqrt{2}}(|+I\rangle - |-I\rangle) \end{aligned}$$

and we find that the two $-i$'s on the state $|1\rangle$ interchange the states undergoing constructive inteference and those undergoing destructive interference in the previous computation. We get

$$\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) = \frac{1}{2}(|+,+I,-I\rangle + |+,-I,+I\rangle + |-,+I,+I\rangle + |-,-I,-I\rangle)$$

Thus, we see the four states with an odd number of $|-\rangle$ and $|-I\rangle$'s, each with probability $\frac{1}{4}$.

Note that the eigenvalues of σ_y are $|\pm I\rangle$, so the above calculation shows that the observable $\sigma_x(1)\otimes\sigma_y(2)\otimes\sigma_y(3) = -1$ for the GHZ state. We can also directly calculate

$$\sigma_x(1)\otimes\sigma_y(2)\otimes\sigma_y(3) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and applying it to the GHZ state, we again find the expectation of $\sigma_x(1)\otimes\sigma_y(2)\otimes\sigma_y(3)$ is -1.

Solution to 2: We have

$$A_1A_2A_3A_4 = f_1(x)^2 f_1(y)^2 f_2(x)^2 f_2(y)^2 f_3(x)^2 f_3(x)^2.$$

Since the values of f_i are ± 1, $f_i^2 = 1$, and thus $A_1A_2A_3A_4 = 1$. We thus have that $A_1 = 1$ and $A_2 = A_3 = A_4 = -1$ is impossible.

Solution to 3: In a local realistic theory, the measurement which is chosen to apply to one qubit cannot affect the outcome obtained on another qubit. Thus, suppose we measure $\sigma_x(1)$, $\sigma_x(2)$, and $\sigma_x(3)$ basis and obtain $+1$ for all three outcomes. Since the expectation of $\sigma_x(1)\otimes\sigma_y(2)\otimes\sigma_y(3)$ is -1, if we had measured the second and third qubits in the $|\pm I\rangle$ basis, we would have had to obtain different outcomes for these measurements. A similar argument shows we have to have obtain different outcomes for the first and second qubits, and for the first and third qubits, had we measured them in the $|\pm I\rangle$ basis. However, if we measure all three in the $|\pm I\rangle$ basis, we cannot obtain different outcomes for each of the three pairs, a contradiction.