In today's class, we will:

- Analyze von Neumann's algorithm in terms of the condition number [this work is due to Dantzig and Eppelman-Freund]

- Do a smoothed analysis of the condition number

1 von Neumann's Linear Programming Algorithm

We consider the following formulation of linear programming:

Given $c, a_1, a_2, \ldots, a_n \in \Re^d$

Is $c \in hull\{a_1, a_2, \ldots, a_n\}$?

In other words, does there exist $x_1, x_2, \ldots, x_n$ such that:

$$\sum a_i x_i = c$$

$$\sum x_i = 1$$

$$x_i \geq 0$$

We would like to find either an x s.t $\|c - \sum a_i x_i\| < \epsilon$ if the linear program is feasible or find a hyperplane separating c from $a_1, a_2, \ldots, a_n$ if it is not.

For technical reasons, we'll assume that the origin lies within the hull.

Recall von Neumann's algorithm for this program.

1. Let $y = Ax$

2. Choose i maximizing $\langle c - y, a_i - y \rangle$

3. If $\langle c - y, a_i - y \rangle / \|c - y\| < \|c - y\|$ then return infeasible

4. Otherwise set x' s.t. $y' = Ax'$ is the point on the line $\overline{ya_i}$ closest to c. Repeat.

The correctness of the algorithm derives from the following:

Claim: If $y, c \in hull\{a_1, a_2, \ldots, a_n\}$ then $\exists i$ s.t.

$$\langle \frac{c-y}{\|c-y\|}, a_i - y \rangle \geq \|c - y\|$$

2   Analysis of the algorithm

Let $r = \text{dist}(c, \text{bdry}(\text{ch}\{a_1, a_2, \ldots, a_n\}))$

Let $R = \max_i \|a_i\|$

Theorem 1 (Eppelman-Freund)
(I) If the linear program is feasible, the algorithm finds x s.t. $\|Ax - c\| \leq \epsilon$ in at most $8(R/r)^2 ln(2R/\epsilon)$ iterations.
(II) If infeasible, the algorithm discovers this in at most $4(R/r)^2$ iterations.

Proof

Part (I)

Claim: If the linear program is feasible then

$$\|Ax' - c\| \leq \|Ax - c\| \sqrt{1 - (\frac{r}{2R})^2}$$

Before proving this claim, let's see why it implies (I). Observe that

$$\sqrt{1 - (\frac{r}{2R})^2} \leq \sqrt{e^{-(r/2R)^2}}$$

Initially,

$$\|Ax_0 - c\| \leq 2R$$

Therefore after k iterations,

$$\|Ax_k - c\| \leq 2R\sqrt{e^{-k(r/2R)^2}}$$

If we let $2R\sqrt{e^{-k(r/2R)^2}} < \epsilon$ we get (I).

Proof of claim

¿From Fig 1. it is clear that,
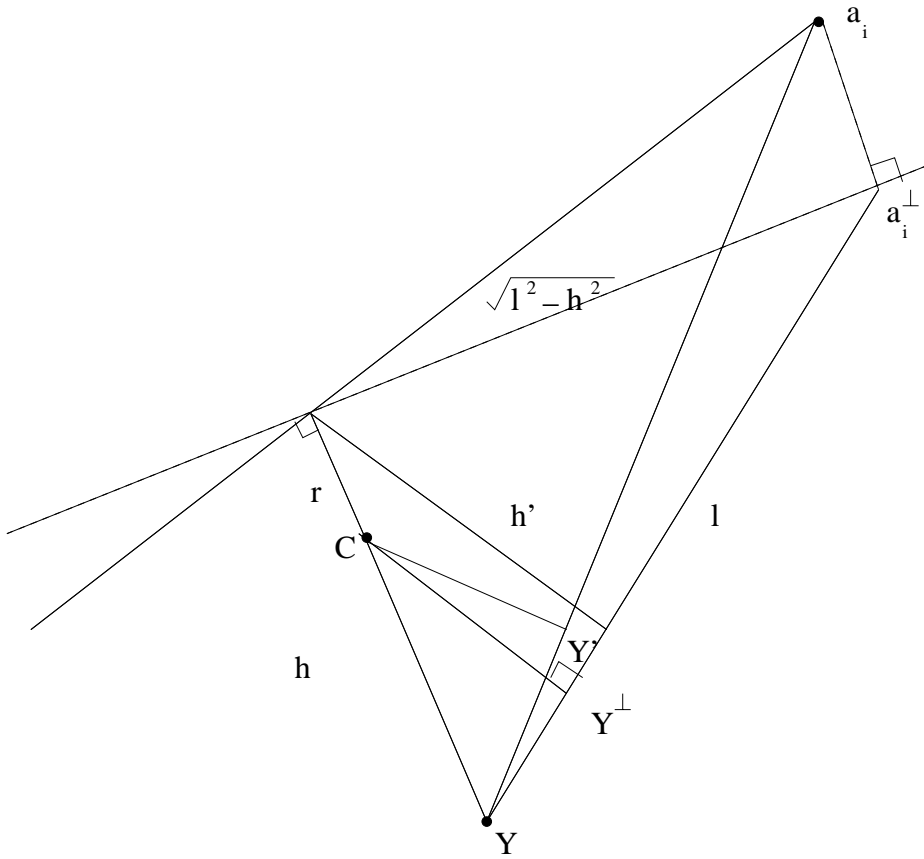$$\text{dist}(c, y') \leq \text{dist}(c, y^\perp)$$

Figure 1: Getting a bound on $dist(c, y')$

By similarity of triangles,

$$\frac{\text{dist}(c, y^{\perp})}{\text{dist}(c, y)} = \frac{h'}{h}$$

Therefore,

$$\begin{aligned}
\frac{dist(c, y')}{dist(c, y)} &\leq \frac{h'}{h} \\
&= \frac{\sqrt{l^2 - h^2}}{l} \\
&= \sqrt{1 - (h/l)^2}
\end{aligned}$$

Since, $h \geq r$ and $l \leq 2R$, it follows that

$$\frac{dist(c, y')}{dist(c, y)} \leq \sqrt{1 - (\frac{r}{2R})^2}$$

<u>Part (II)</u>
Claim (Dantzig) After k iterations,

$$\|Ax - c\| \le \frac{2R}{\sqrt{k}}$$

Again it's not hard to show why this implies (II). Since point y cannot go outside the convex hull, the algorithm will stop once

$$\frac{2R}{\sqrt{k}} \le r$$

So the algorithm must terminate after $(2R/r)^2$ iterations.

Proof of claim

Let h_k be the height (i.e., distance between c and y) after k^{th} iteration. As in the previous case, we can show that,

$$h_{k+1} \le h_k \sqrt{1 - (h_k/2R)^2}$$

We claim that

$$h_k \le 2R/\sqrt{k}$$

Proof by induction:

Note that $\|c\| \le R$ since otherwise it would be trivial to decide infeasibility.

Therefore, $h_1 \le 2R$

Let $h_k \le 2R/\sqrt{k}$

Then

$$
\begin{aligned}
h_{k+1} &\le h_k \sqrt{1 - (h_k/2R)^2} \\
&= 2R/\sqrt{k} \sqrt{1 - 1/k} \\
&= 2R\sqrt{\frac{k-1}{k^2}} \\
&\le 2R/\sqrt{k+1}
\end{aligned}
$$

■

3 Condition number of a linear program

Given a linear program, $c, a_1, a_2, \ldots, a_n$, we define it's condition number, K, as follows:

If $c \in hull\{a_1, a_2, \ldots, a_n\}$,

$K(c, a_1, a_2, \ldots, a_n) = inf\{\|\Delta c\| + \sum_i \|a_i\| : c + \Delta c \notin hull\{a_1 + \Delta a_1, a_2 + \Delta a_2, \ldots, a_n + \Delta a_n\}\}$

In other words, K measures the smallest change to infeasibility. Similarly,

If $c \notin hull\{a_1, a_2, \ldots, a_n\}$,

$K(c, a_1, a_2, \ldots, a_n) = inf\{\|\Delta c\| + \sum_i \|a_i\| : c + \Delta c \in hull\{a_1 + \Delta a_1, a_2 + \Delta a_2, \ldots, a_n + \Delta a_n\}\}$

Claim 2 $K(c, a_1, a_2, \ldots, a_n) = r$

It is obvious that $K(c, a_1, a_2, \ldots, a_n) \leq r$

We'll try to prove $K(c, a_1, a_2, \ldots, a_n) \geq r$

Lemma 3 *Let* $C = hull\{a_1, a_2, \ldots, a_n\}, C' = hull\{a_1 + \Delta a_1, a_2 + \Delta a_2, \ldots, a_n + \Delta a_n\}$

1. $\gamma(C, C') \leq \max_i \|\Delta a_i\|$

2. $\gamma(\bar{C}, \bar{C}') \leq \max_i \|\Delta a_i\|$

3. $\gamma(bdry(C), bdry(C')) \leq \max_i \|\Delta a_i\|$

where $\gamma(A, B) = \max_{x \in A} \min_{y \in B} dist(x, y)$

Proof

1. Let $x \in C$
   $\Rightarrow \exists \alpha_1, \alpha_2, \ldots, \alpha_n$ s.t.

$$\sum \alpha_i = 1$$
$$\alpha_i \geq 0$$
$$x = \sum_i \alpha_i a_i$$

Let $x' = \sum \alpha_i (a_i + \Delta a_i) \in C'$. Then,

$$dist(x, x') = \|\sum \alpha_i \Delta a_i\|$$
$$\leq (\sum \alpha_i) \max_i \|\Delta a_i\|$$
$$= \max_i \|\Delta a_i\|$$

5

2. This is similar to 1.

3. Let $x \in bdry(C)$

   $\exists x_1 \in C' \, s.t. \, dist(x, x_1) \le \max_i \|\Delta a_i\|$

   Similarly, $\exists x_2 \in \bar{C}' \, s.t. \, dist(x, x_2) \le \max_i \|\Delta a_i\|$

   Therefore, on the line from x_1 to x_2 $\exists$ a point on $bdry(C')$ with distance from x at most $\max_i \|\Delta a_i\|$

■

It takes a little more work to actually use this lemma to prove the fact that $K \ge r$. Intuitively, what it means is that the boundary of the convex hull doesn't move much by changing the a_i's. Therefore, it is better to just change c.

In the next lecture, John Dunagan will do a smoothed analysis of the condition number. In particular, we shall prove the following theorem.

Theorem 4 *For $c, a_1, a_2, \ldots, a_n$ Gaussian random vectors with variance σ^2 and centered at $\bar{c}, \bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n$, such that each has norm ≤ 1,*

$$Pr[K(c, a_1, a_2, \ldots, a_n) < \epsilon] \le \frac{128 d^{1/2} \epsilon}{\sigma}$$

This result actually follows trivially from an earlier result, but with a less intuitive proof.

Theorem 5 (Keith Ball '93) *For any convex body K, and Gaussian random vector c,*

$$Pr[dist(c, bdry(K)) < \epsilon] \le \frac{8 d^{1/4} \epsilon}{\sigma}$$

6