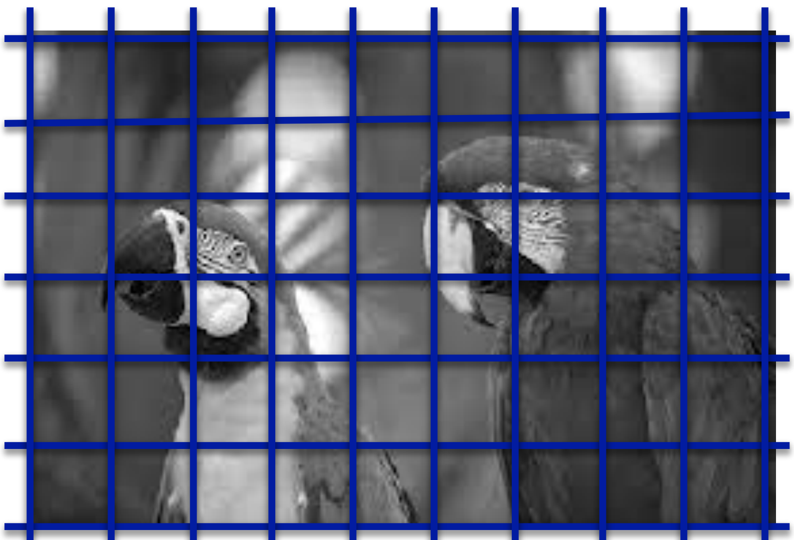
# Simple, Efficient and Neural Algorithms for Sparse Coding

Ankur Moitra (MIT)

joint work with Sanjeev Arora, Rong Ge and Tengyu Ma

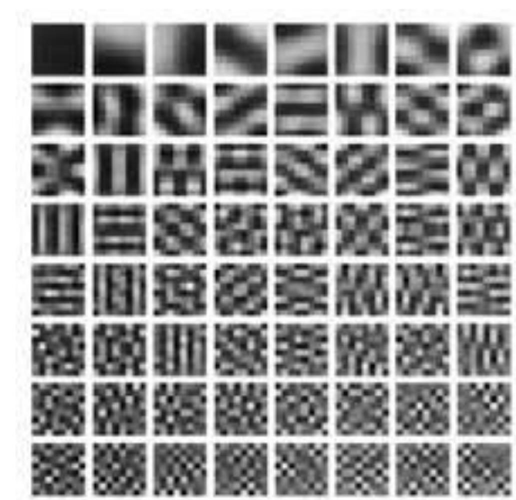
B. A. Olshausen, D. J. Field. "Emergence of simple-cell receptive field properties by learning a sparse code for natural images", 1996

break **natural images** into patches:



(collection of vectors)

**sparse coding**





**Properties:** localized, bandpass and oriented

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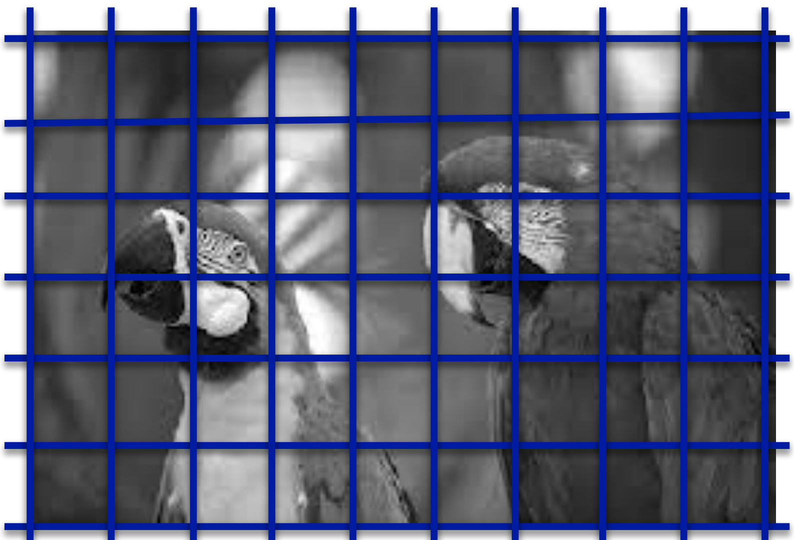


**singular value decomposition**

Noisy!

Difficult to interpret!

(collection of vectors)

# OUTLINE

Are there efficient, neural algorithms for sparse coding with **provable guarantees**?

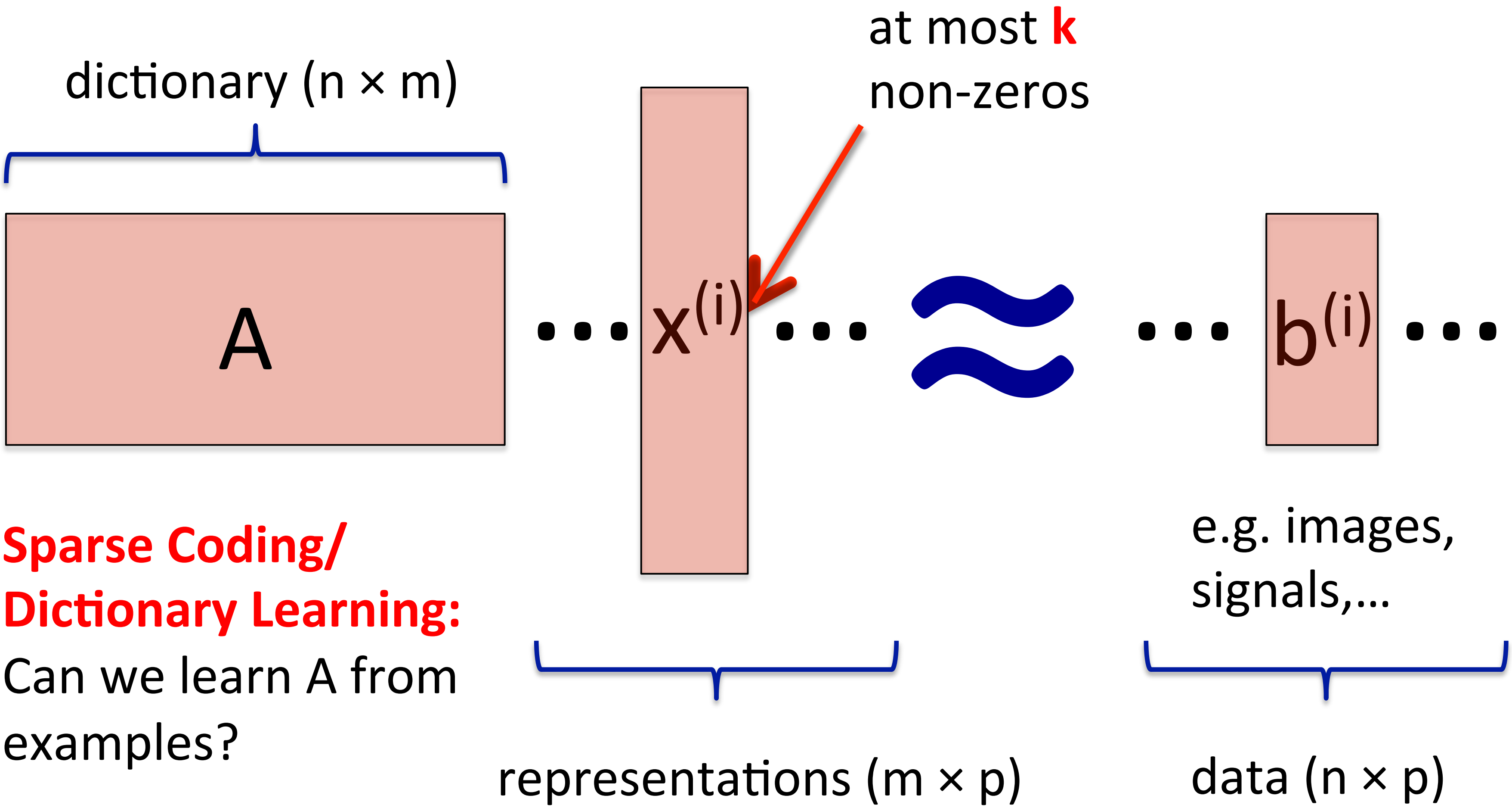
## Part I: The Olshausen-Field Update Rule

- A Non-convex Formulation
- Neural Implementation
- A Generative Model; Prior Work

## Part II: A New Update Rule

- Online, Local and Hebbian with Provable Guarantees
- Connections to Approximate Gradient Descent
- Further Extensions

More generally, many types of data are sparse in an appropriately chosen basis:




**Sparse Coding/ Dictionary Learning:** Can we learn A from examples?

# NONCONVEX FORMULATIONS

Usual approach, minimize **reconstruction error**:

$$\min_{A,\, x^{(i)}\text{'s}} \sum_{i=1}^{p} \| b^{(i)} - A\, x^{(i)} \| + \sum_{i=1}^{p} L(x^{(i)})$$

**non-linear penalty function**

(encourage sparsity)

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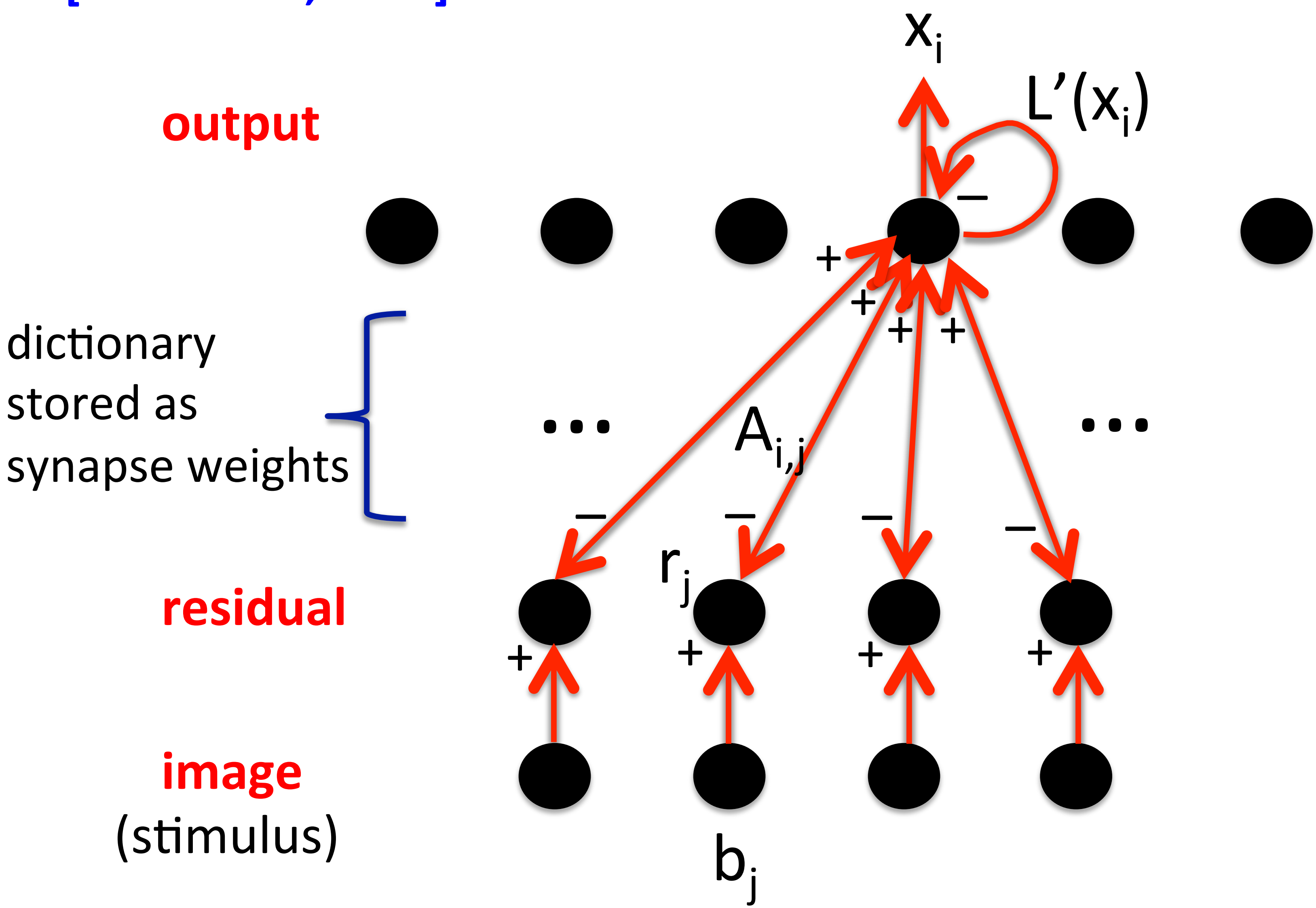
This optimization problem is **NP-hard**, can have many local optima; but **heuristics** work well nevertheless…

# A NEURAL IMPLEMENTATION

**[Olshausen, Field]:**



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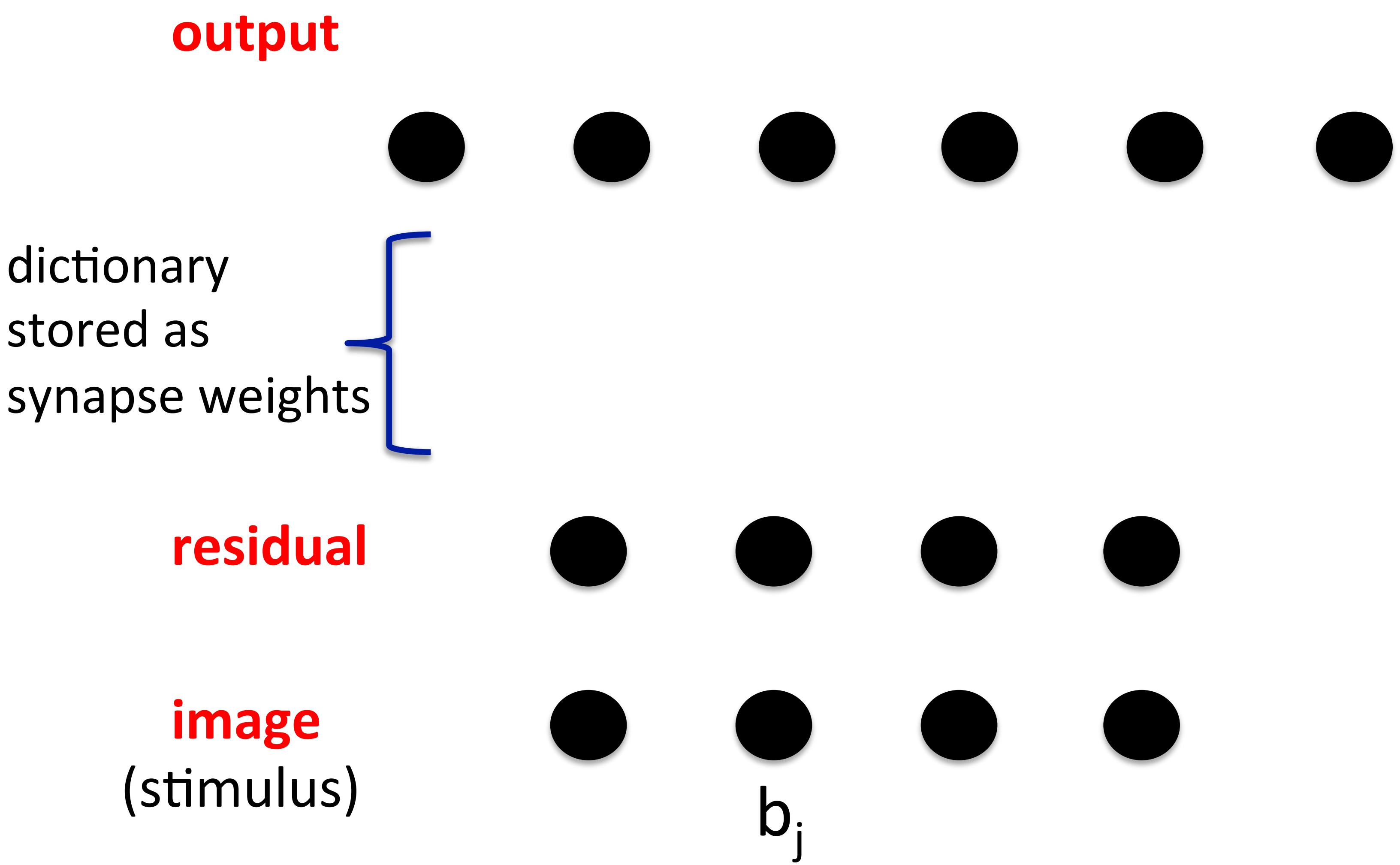

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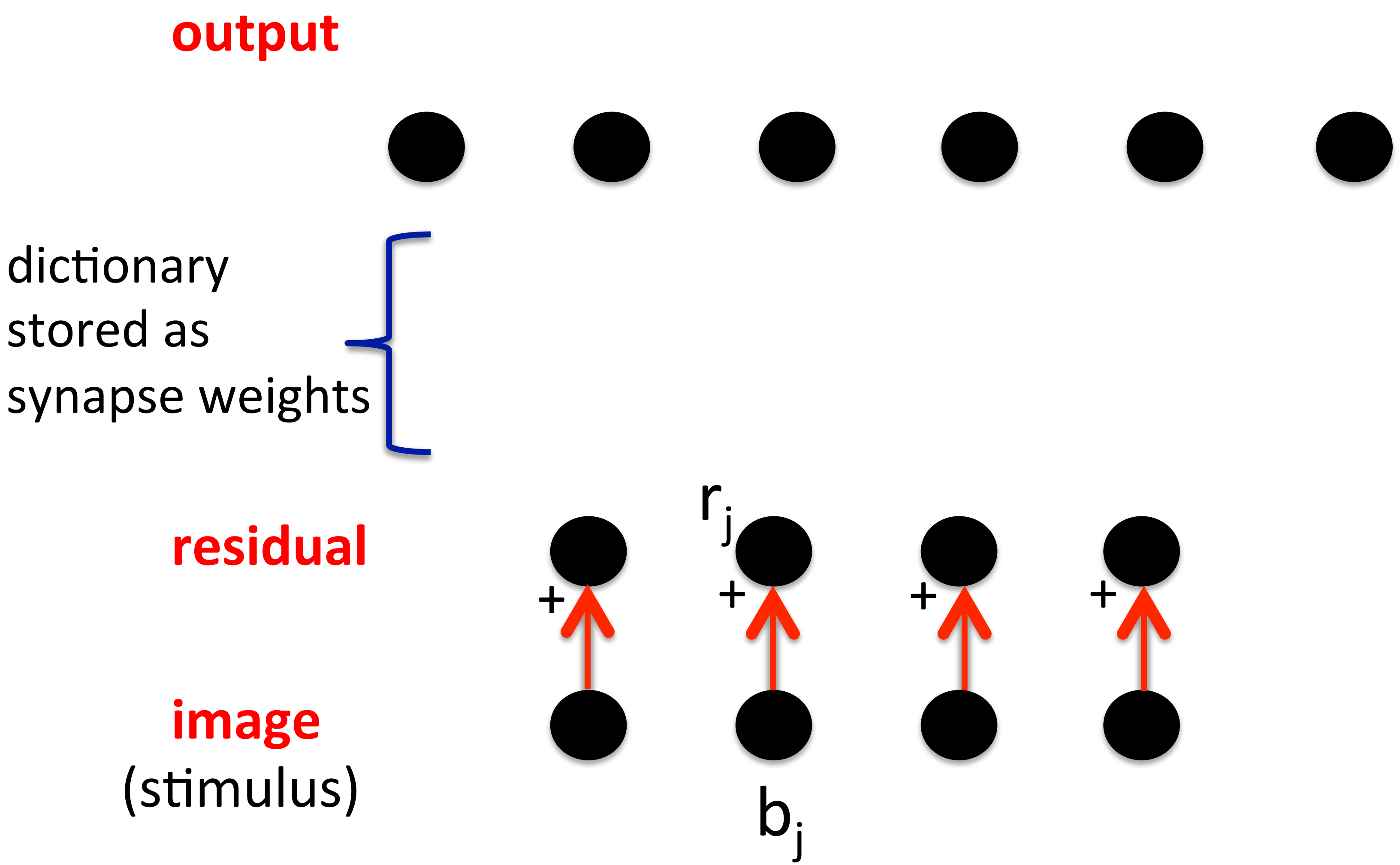

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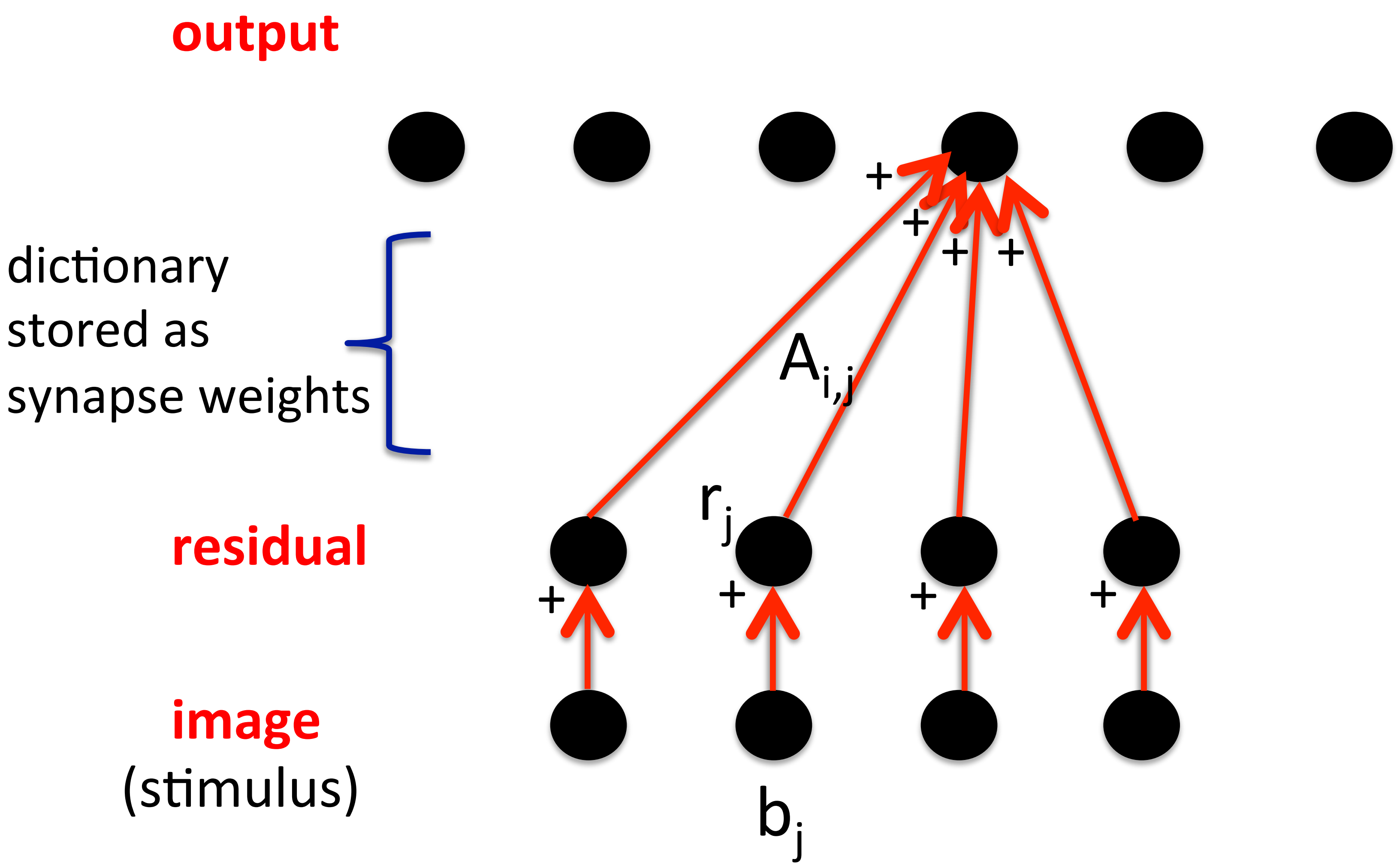

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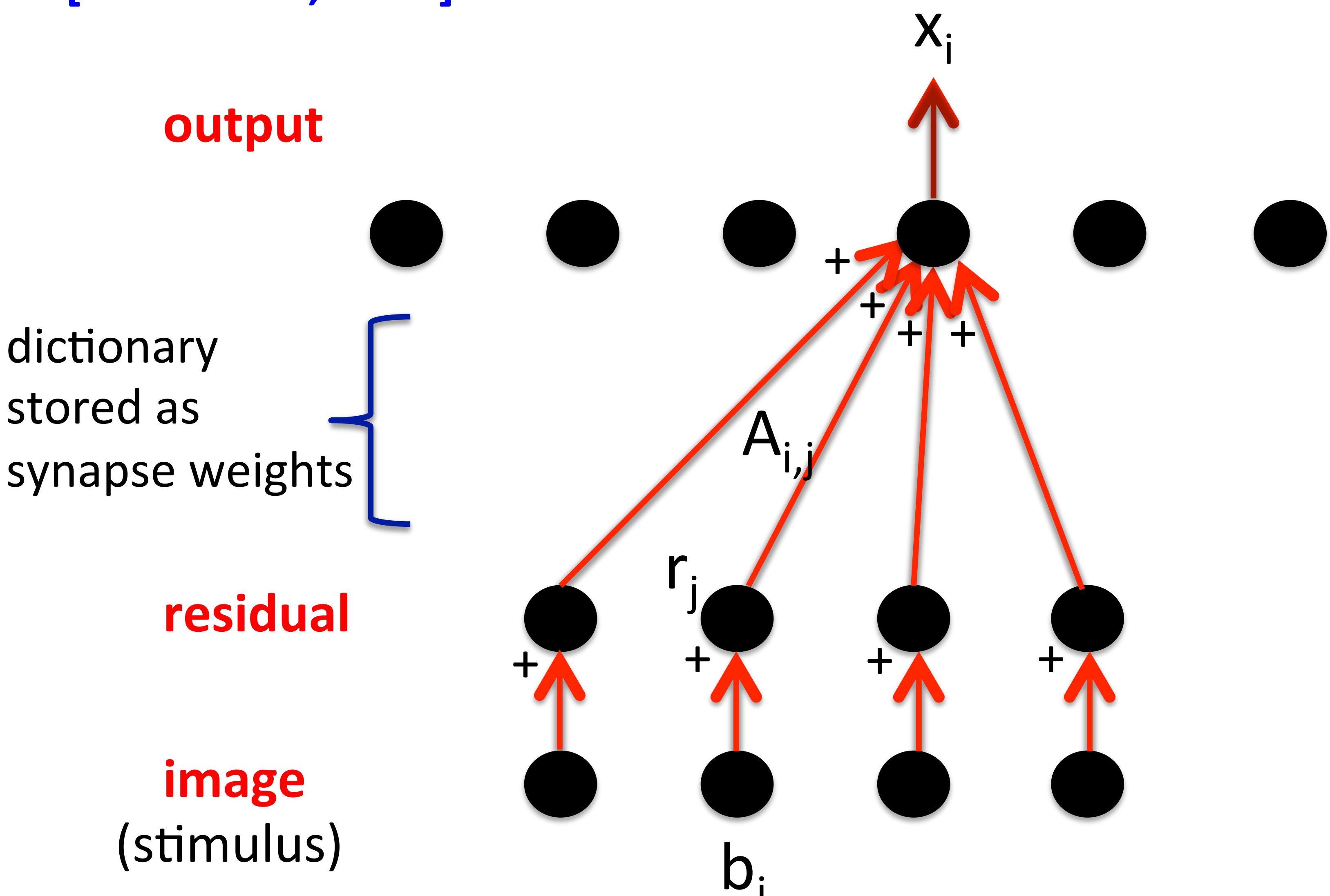

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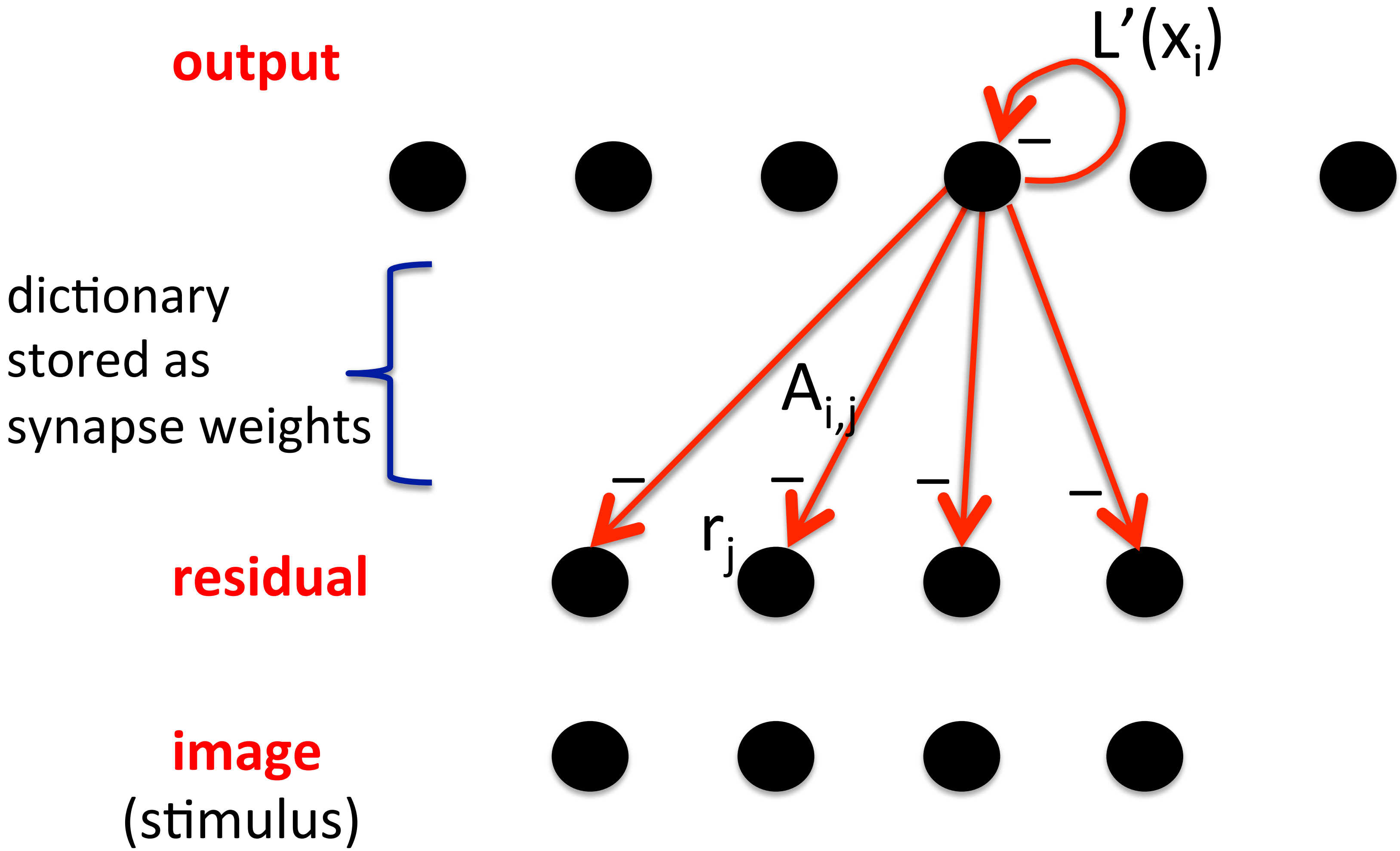

This network performs **gradient descent** on:

$$\|b - A x\|^2 + L(x)$$

by alternating between **(1)** $r \leftarrow b - Ax$

**(2)** $x \leftarrow x + \eta(A^T r - \nabla L(x))$

Moreover A is updated through **Hebbian rules**

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There are no **provable guarantees**, but works well

But **why** should gradient descent on a non-convex function work?

Are simple, local and Hebbian rules sufficient to find **globally** optimal solutions?

# OTHER APPROACHES, AND APPLICATIONS

**Signal Processing/Statistics (MOD, kSVD):**

- De-noising, edge-detection, super-resolution
- Block compression for images/video

**Machine Learning (LBRN06, …):**

- Sparsity as a **regularizer** to prevent over-fitting
- Learned sparse reps. play a key role in deep-learning

**Theoretical Computer Science (SWW13, AGM14, AAJNT14):**

- New algorithms with **provable** guarantees, in a natural **generative model**

**Generative Model**:

- unknown dictionary A
- generate x with support of size k u.a.r., choose non-zero values independently, observe b = Ax

**[Spielman, Wang, Wright '13]:** works for full coln rank A up to sparsity roughly $n^{1/2}$ **(hence $m \leq n$)**

**[Arora, Ge, Moitra '14]:** works for overcomplete, μ-incoherent A up to sparsity roughly $n^{1/2-\varepsilon}/\mu$

**[Agarwal et al. '14]:** works for overcomplete, μ-incoherent A up to sparsity roughly $n^{1/4}/\mu$, via alternating minimization

**[Barak, Kelner, Steurer '14]:** works for overcomplete A up to sparsity roughly $n^{1-\varepsilon}$, but running time is **exponential** in accuracy

# OUR RESULTS

Suppose $k \leq \sqrt{n}/\mu$ polylog(n) and $||A|| \leq \sqrt{n}$ polylog(n)

Suppose $\widehat{A}$ that is column-wise $\delta$-close to A for $\delta \leq 1/\text{polylog}(n)$

**Theorem [Arora, Ge, Ma, Moitra '14]:** There is a variant of the OF-update rule that converges to the true dictionary at a **geometric rate**, and uses a polynomial number of samples

All previous algorithms had suboptimal sparsity, worked in less generality, or were **exponential** in a natural parameter

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**Note:** $k \leq \sqrt{n}/2\mu$ is a barrier, even for sparse recovery

i.e. if $k > \sqrt{n}/2\mu$, then x is not necessarily the sparsest soln to Ax = b

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# A NEW UPDATE RULE

Alternate between the following steps (size q batches):

**(1)** $\hat{x}^{(i)} = \text{threshold}(\hat{A}^{T} b^{(i)})$

**(2)** $\hat{A} \leftarrow \hat{A} + \eta \sum_{i=1}^{q} (b^{(i)} - \hat{A}\hat{x}^{(i)})\,\text{sgn}(\hat{x}^{(i)})^{T}$

# A NEW UPDATE RULE

Alternate between the following steps (size q batches):

**(1)** $\hat{x}^{(i)} = \text{threshold}(\hat{A}^T b^{(i)})$ (zero out small entries)

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In contrast, previous (provable) algorithms might need to compute a new estimate **from scratch**, when new samples arrive

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In particular, the output is a thresholded, weighted sum of activations

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"neurons that fire together, wire together"

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The update rule is explicitly **Hebbian**

The update to a weight $\hat{A}_{i,j}$ is the product of the activations at the residual layer and the decoding layer

# WHAT IS NEURALLY PLAUSIBLE, ANYWAYS?

Our update rule (essentially) inherits a neural implementation from **[Olshausen, Field]**

However there are many competing theories for what constitutes a **plausible** neural implementation

e.g. nonnegative outputs, no bidirectional links, etc…

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But ours is **online**, **local** and **Hebbian**, all of which are basic properties to require

The surprise is that such simple building blocks can find **globally optimal** solutions to **highly non-trivial** algorithmic problems!

# APPROXIMATE GRADIENT DESCENT

We give a general framework for **designing** and **analyzing** iterative algorithms for sparse coding

The usual approach is to think of them as trying to minimize a **non-convex** function:

$$\min_{\widehat{A},\ \text{coln-sparse}\ \widehat{X}} \quad E(\widehat{A}, \widehat{X}) = \left\| B - \widehat{A}\,\widehat{X} \right\|_F^2$$

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colns are $b^{(i)}$'s

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# APPROXIMATE GRADIENT DESCENT

We give a general framework for **designing** and **analyzing** iterative algorithms for sparse coding

How about thinking of them as trying to minimize an **unknown**, **convex** function?

$$\min_{\hat{A}} \quad E(\hat{A}, X) = \left\| B - \hat{A} X \right\|_F^2$$

Now the function is strongly convex, and has a global optimum that can be reached by gradient descent!

**New Goal:** Prove that (with high probability) the step **(2)** approximates the gradient of this function

# CONDITIONS FOR CONVERGENCE

Consider the following general setup:

**optimal solution:** $z^*$

**update:** $z^{s+1} = z^s - \eta\, g^s$

**Definition:** $g^s$ is $(\alpha, \beta, \varepsilon_s)$-correlated with $z^*$ if for all s:

$$\langle g^s, z^s - z^* \rangle \geq \alpha \| z^s - z^* \|^2 + \beta \| g^s \|^2 - \varepsilon_s$$

**Theorem:** If $g^s$ is $(\alpha, \beta, \varepsilon_s)$-correlated with $z^*$, then

$$\| z^s - z^* \|^2 \leq (1 - 2\alpha\eta)^s \| z^0 - z^* \|^2 + \frac{\max_s \varepsilon_s}{\alpha}$$

This follows immediately from the usual proof…

$$\textbf{(1)} \quad \hat{x}^{(i)} = \text{threshold}(\hat{A}^T b^{(i)})$$

**Decoding Lemma:** If $\hat{A}$ is 1/polylog(n)-close to A and $\|\hat{A} - A\| \leq 2$, then decoding recovers the signs correctly (whp)

$$\textbf{(2)} \quad \hat{A} \leftarrow \hat{A} + \eta \sum_{i=1}^{q} (b^{(i)} - \hat{A}\hat{x}^{(i)})\,\text{sgn}(\hat{x}^{(i)})^T$$

**Key Lemma:** Expectation of (the column-wise) update rule is

$$\hat{A}_j \leftarrow \hat{A}_j + \xi\,(I - \hat{A}_j\hat{A}_j^T)A_j + \xi\,\mathbf{E}_R[\hat{A}_R\hat{A}_R^T]A_j + \text{error}$$

$A_j - \hat{A}_j$

systemic bias

where R = supp(x)\j, if decoding recovers the correct signs

**Auxiliary Lemma:** $\|\hat{A} - A\| \leq 2$, remains true throughout if η is small enough and q is large enough

**Proof:** Let $\zeta$ denote any vector whose norm is negligible (say, $n^{-\omega(1)}$).

$$g_j = E[(b - \hat{A}\hat{x})\mathrm{sgn}(\hat{x}_j)]$$

is the expected update to $\hat{A}_j$. Let $1_F$ be the indicator of the event that decoding recovers the signs of x.

$$g_j = E[(b - \hat{A}\hat{x})\mathrm{sgn}(\hat{x}_j)\, 1_F] + E[(b - \hat{A}\hat{x})\mathrm{sgn}(\hat{x}_j)\, 1_{\bar{F}}]$$

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$$g_j = E[(b - \hat{A}\,\mathrm{threshold}(\hat{A}^T b))\ \mathrm{sgn}(x_j)\ 1_F] \pm \zeta$$

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$$g_j = E[(b - \widehat{A}_S \widehat{A}_S^T b)\ \mathrm{sgn}(x_j)\ 1_F] \pm \zeta$$

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$$g_j = E[(I - \widehat{A}_S\widehat{A}_S^T)Ax\ \mathrm{sgn}(x_j)] \pm \zeta$$

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$$= p_j q_j\ (I - \widehat{A}_j\widehat{A}_j^T)A_j + p_j\ \widehat{A}_{-j}\text{diag}(q_{i,j})\widehat{A}_{-j}^T\ A_j \pm \zeta$$

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# AN INITIALIZATION PROCEDURE

We give an initialization algorithm that outputs $\widehat{A}$ that is column-wise $\delta$-close to A for $\delta \leq 1/\text{polylog}(n)$, $\|\widehat{A} - A\| \leq 2$

**Repeat:** **(1)** Choose samples b, b′

**(2)** Set $M_{b,b'} = \frac{1}{q}\sum_{i=1}^{q} (b^T b^{(i)})(b'^T b^{(i)})\, b^{(i)} (b^{(i)})^T$

**(3)** If $\lambda_1(M_{b,b'}) > \frac{k}{m}$ and $\lambda_2 << \frac{k}{m \log m}$

output top eigenvector

# AN INITIALIZATION PROCEDURE

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**(3)** If $\lambda_1(M_{b,b'}) > \frac{k}{m}$ and $\lambda_2 \ll \frac{k}{m \log m}$

output top eigenvector

**Key Lemma:** If Ax = b and Ax′ = b′, then condition **(3)** is satisfied if and only if $\text{supp}(x) \cap \text{supp}(x') = \{j\}$ in which case, the top eigenvector is δ-close to $A_j$

# DISCUSSION

Our initialization gets us to $\delta \leq 1/\text{polylog}(n)$, can be neurally implemented with **Oja's Rule**

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Earlier analyses of alternating minimization for $\delta \leq 1/\text{poly}(n)$ in **[Arora, Ge, Moitra '14]** and **[Agarwal et al '14]**

However, in those settings A and $\hat{A}$ are so close that the objective function is **essentially convex**

We show that it converges even from **mild** starting conditions

As a result, our bounds improve on existing algorithms in terms of **running time**, **sample complexity** and **sparsity** (all but SOS)

# FURTHER RESULTS

Adjusting an iterative alg. can have subtle effects on its behavior

We can use our framework to **systematically** design/analyze new update rules

E.g. we can remove the **systemic bias**, by carefully projecting out along the direction being updated

**(1)** $$\hat{x}_j^{(i)} = \text{threshold}(\hat{C}_j^T b^{(i)})$$

where $\hat{C}_j = [\text{Proj}_{\hat{A}_j^\perp}(\hat{A}_1), \text{Proj}_{\hat{A}_j^\perp}(\hat{A}_2), \ldots \hat{A}_j \ldots \text{Proj}_{\hat{A}_j^\perp}(\hat{A}_m)]$

**(2)** $$\hat{A}_j \leftarrow \hat{A}_j + \eta \sum_{i=1}^{q} (b^{(i)} - \hat{C}_j \hat{x}_j^{(i)}) \text{sgn}(\hat{x}_j^{(i)})^T$$

# Any Questions?

**Summary:**

- **Online**, **local** and **Hebbian** algorithms for sparse coding that find a globally optimal solution (whp)

- Introduced a framework for analyzing iterative algorithms by thinking of them as trying to minimize an **unknown**, **convex** function

- The key is working with a generative model

- Is **computational intractability** really a barrier to a rigorous theory of neural computation?

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