6.841: Advanced Complexity Theory	Spring 2016

Problem Set 3

Due Date: April 5th, 2016

Turn in your solution to each problem on a separate piece of paper. Mark the top of each sheet with the following: (1) your name, (2) the question number, (3) the names of any people you worked with on the problem, or "Collaborators: none" if you solved the problem individually. We encourage you to spend time on each problem individually before collaborating!

Problem 1 – Log Rank: Another Technique Lower Bounding Deterministic CC

In the problems below, let $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ denote a boolean function. Let M_f denote the *communication matrix* of f, where the rows and columns of M_f are indexed by strings in $\{0,1\}^n$ and $M_f[x,y] = f(x,y)$.

a) Prove that $D(f) \geq \log(\text{rank}(M_f))$, where the rank is taken over the reals. Prove the same result when the rank is taken over $\mathbb{F}_2$.

b) Prove that if all rows of M_f are distinct, then $D(f) \geq \log n$.

c) Prove that $D(f) \leq \text{rank}(M_f) + 1$, where the rank is over the reals.

d) Show that $D(f) \leq O(N^1(f) \log \text{rank}(M_f))$, where the rank is over the reals.

   Hint: Look at the combinatorial proof of Aho-Ullman-Yannakakis posted on Piazza.

Problem 2 – Randomized Protocol for GT

Let

$$GT(x,y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{otherwise} \end{cases}$$

Prove that $R^{pub}(GT) \leq O(\log n \log\log n)$. Conclude that $R(GT) \leq O(\log n \log\log n)$ by Newman's Theorem.

Problem 3 – Balancing Protocol Trees

If f has a deterministic communication protocol tree with ℓ leaves, show that f has a deterministic communication protocol tree with depth $O(\log \ell)$.
It may be helpful to use the following fact: given a binary tree with ℓ leaves, there exists a subtree that has at least $\frac{\ell}{3}$ leaves and at most $\frac{2\ell}{3}$ leaves. If you use this fact, you should include a proof of it.

Problem 4 – Communication Protocol for Median

Define the function $\text{MED}(x, y) : \{0,1\}^n \times \{0,1\}^n \to [n] = \{1, \ldots, n\}$ as follows. Interpret the strings x and y as subsets of $[n]$, where a 1 in the string indicates membership in the set. Alice receives x, and Bob receives y, and their task is to compute the median of the multiset $x \cup y$.

a) Show that $D(\text{MED}) \leq O(\log^2 n)$

b) Show that $D(\text{MED}) \leq O(\log n)$. (Hint: Assume both sets are the same size, and show that, with a constant amount of communication, Alice and Bob can either reduce the sizes of their sets by a constant fraction or narrow the range in which their median lies by a constant fraction. Then justify why you can take the sets to be the same size).

Problem 5: Clique versus Independent Set

Given a graph G with n vertices known to both Alice and Bob, define the communication problem CIS_G as follows. Alice receives a clique C in G, and Bob gets an independent set I in G. Their task is to compute $|C \cap I|$ with minimal communication. Note that communicating a single vertex takes $\log n$ bits of communication.

a) Show that $D(\text{CIS}_G) \leq O(\log^2 n)$

b) The *1-partition number* of f, $C_1^D(f)$, is the number of monochromatic rectangles of 1's that required to partition the 1's in the communication matrix M_f (i.e. the rectangles are disjoint). Show that, if $D(\text{CIS}_G) \leq O(\log^c n)$, then for any function f, $D(f) \leq O(\log(C_1^D(f))^c)$. [1]

[1]Until very recently, it was open if there was a protocol for CIS_G that achieved $O(\log^c n)$ communication for $c < 2$. In 2015, Göös, Pitassi and Watson proved that there exists a function f such that $D(f) \geq \tilde{\Omega}(\log^2(C_1^D(f)))$.