

# Bead moving along a thin, rigid, wire.

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## Abstract

An equation describing the motion of a bead along a rigid wire is derived. First the case with no friction is considered, and a Lagrangian formulation is used to derive the equation. Next a simple correction for the effect of friction is added to the equation. Finally, we consider the case where friction dominates over inertia, and use this to reduce the order of the system.

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# 1 Equations with no friction.

Consider the motion of a bead of mass  $M$  moving along a thin rigid wire, under the influence of gravity. Let  $(x, y, z)$  be a system of Cartesian coordinates, with  $z$  the vertical direction and  $z$  increasing with height. Let  $g$  be the acceleration of gravity. Describe the wire in parametric form, as follows:

$$x = X(s, t), \quad y = Y(s, t), \quad \text{and} \quad z = Z(s, t), \tag{1.1}$$

where  $s$  is the arclength along the wire. Note that:

- Because the wire is “thin”, we have approximated it as just a curve in space.
- The wire is allowed to move and change shape as a function of time.
- The bead will be approximated as a mass point.
- We use the same length units for  $x, y, z,$  and  $s,$  so that:  $X_s^2 + Y_s^2 + Z_s^2 = 1.$

Furthermore: we will assume that the wire does not stretch as it moves. In particular,  $s$  is not only arclength, but a material coordinate along the wire (i.e.  $s$  constant denotes always the same material point on the wire). This will be important when we consider the effects of friction.

It is well known that, in the presence of rigid body constraints, the best way to formulate equations in mechanics is by the use of Lagrangian mechanics — that is: by using the principle of least action.

**Remark 1.1** *Of course, one can get the equations using the classical newtonian formulation. What complicates things is that one must consider the “unknown” forces that arise to keep the constraints valid. For example, in this case the bead experiences the force of gravity, and an unknown force from the wire — which must be exactly right to keep the bead moving along the wire (this is what determines this force). When the wire is static, calculating this force is relatively easy. However, when the wire moves, things get complicated.*

**Remark 1.2 (Principle of least action).** *In its simplest form, the principle of least action for a conservative system (no energy is dissipated) can be stated as follows: Let  $\Gamma$  be a mechanical trajectory, joining the points  $P_1$  and  $P_2$  in phase space. Parameterize this trajectory by time  $t$ , with  $t = t_1$  the starting time (when the trajectory is at  $P_1$ ), and  $t = t_2$  the final time (when  $P_2$  is reached). Consider now all the possible trajectories (curves) in phase space, starting at  $P_1$  for  $t = t_1$ , and ending at  $P_2$  for  $t = t_2$  —  $\Gamma$  is, of course, one member of this set. For any each such curve the action  $\mathcal{L}$  can be computed:*

$$\mathcal{L} = \int_{t_1}^{t_2} (\text{Kinetic Energy} - \text{Potential Energy}) dt.$$

Then the principle of least action says that  $\Gamma$  is a stationary point for  $\mathcal{L}$  within the set of all possible trajectories described above.

We now apply this principle to our problem. Let the position of the bead along the wire be given by  $s = \mathcal{S}(t)$ . Then:

$$\begin{aligned} \text{Kinetic Energy} &= \frac{1}{2} M (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\ &= \frac{1}{2} M \dot{\mathcal{S}}^2 + M (X_s X_t + Y_s Y_t + Z_s Z_t) \dot{\mathcal{S}} + \frac{1}{2} M (X_t^2 + Y_t^2 + Z_t^2) \\ &= \frac{1}{2} M \dot{\mathcal{S}}^2 + \kappa \dot{\mathcal{S}} + \beta, \end{aligned} \tag{1.2}$$

$$\text{Potential Energy} = g M Z, \tag{1.3}$$

where: (A)  $\kappa = \kappa(\mathcal{S}, t) = M (X_s X_t + Y_s Y_t + Z_s Z_t)$  and  $\beta = \beta(\mathcal{S}, t) = \frac{1}{2} M (X_t^2 + Y_t^2 + Z_t^2)$ .

(B)  $X, Y, Z$ , and all their derivatives are evaluated at  $(\mathcal{S}, t)$ .

(C) The dots indicate (total) derivatives with respect to time.

(D) Since  $x = X(\mathcal{S}, t)$ , we have  $\dot{x} = \dot{\mathcal{S}} X_s(\mathcal{S}, t) + X_t(\mathcal{S}, t)$

— with similar formulas for  $\dot{y}$  and  $\dot{z}$ .

(E) We have used that  $s$  is arclength, so that  $X_s^2 + Y_s^2 + Z_s^2 = 1$ .

The action is then given by

$$\mathcal{L} = \mathcal{L}[\mathcal{S}] = \int_{t_1}^{t_2} L(\dot{\mathcal{S}}, \mathcal{S}, t) dt, \tag{1.4}$$

where  $L = \frac{1}{2} M \dot{\mathcal{S}}^2 + \kappa \dot{\mathcal{S}} - V$ , with  $V = V(\mathcal{S}, t) = g M Z - \beta$ .

Because of the principle of least action, the *Euler-Lagrange* equation

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\mathcal{S}}} \right) - \frac{\partial L}{\partial \mathcal{S}} = 0, \quad (1.5)$$

must apply for the mechanical trajectory. Using the expression for  $L$  above, this yields the equation:

$$M \frac{d^2 \mathcal{S}}{dt^2} = -\frac{\partial V}{\partial \mathcal{S}}(\mathcal{S}, t) - \frac{\partial \kappa}{\partial t}(\mathcal{S}, t). \quad (1.6)$$

This is the **equation for the motion of a bead along a rigid moving wire, under the action of gravity, when friction is neglected.**

**Example 1.1** When the wire moves normal to itself,  $\kappa \equiv 0$ . *An example is a hoop rotating around a diameter.* Consider now the case with vertical rotation axis:

$$\begin{aligned} X &= +R \sin\left(\frac{s}{R}\right) \cos(\Omega t), \\ Y &= +R \sin\left(\frac{s}{R}\right) \sin(\Omega t), \\ Z &= -R \cos\left(\frac{s}{R}\right), \end{aligned}$$

where  $R$  is the radius of the hoop and  $\Omega$  is its angular velocity. Then

$$\kappa = 0 \quad \text{and} \quad V = -gMR \cos\left(\frac{\mathcal{S}}{R}\right) - \frac{1}{2}MR^2\Omega^2 \sin^2\left(\frac{\mathcal{S}}{R}\right), \quad (1.7)$$

where we **notice that  $V$  is independent of time.** The equation of motion is then:

$$\frac{d^2 \mathcal{S}}{dt^2} = -g \sin\left(\frac{\mathcal{S}}{R}\right) + \frac{1}{2}R\Omega^2 \sin\left(\frac{2\mathcal{S}}{R}\right). \quad (1.8)$$

Rewrite the equation using the angle  $\Theta = \frac{\mathcal{S}}{R}$  that the bead makes with the vertical down-radius:

$$\frac{d^2 \Theta}{dt^2} = -\omega^2 \sin \Theta + \frac{1}{2} \Omega^2 \sin 2\Theta = (\Omega^2 \cos \Theta - \omega^2) \sin \Theta, \quad (1.9)$$

where  $\omega = \sqrt{g/R}$  is the angular frequency for a pendulum of length  $R$ . The two terms on the right in equations (1.8 – 1.9) correspond to the effects of gravity and the centrifugal force.

**Problem 1.1** Consider the case of a hoop rotating around an horizontal diameter.

**Example 1.2** A generalization of example 1.1 is that of a wire on a vertical plane, rotating around a vertical axis. Then:

$$\begin{aligned} X &= R(s) \cos(\Omega t), \\ Y &= R(s) \sin(\Omega t), \\ Z &= Z(s), \end{aligned}$$

where  $(R')^2 + (Z')^2 = 1$ , and  $\Omega$  is the rotation angular velocity. Then

$$\kappa = 0 \quad \text{and} \quad V = gMZ(\mathcal{S}) - \frac{1}{2}M\Omega^2 R^2(\mathcal{S}), \quad (1.10)$$

where, again,  $V$  is independent of time. The equation of motion is then

$$\frac{d^2\mathcal{S}}{dt^2} + gZ'(\mathcal{S}) - \Omega^2 R(\mathcal{S})R'(\mathcal{S}) = 0. \quad (1.11)$$

**Problem 1.2** Generalize the derivations in examples 1.1 and 1.2 to consider variable rotation rates. That is to say: in the formulas describing the wires, replace  $\Omega t \rightarrow \Psi$ , where  $\Psi = \Psi(t)$  is some function of time. Define  $\Omega = \Omega(t) = d\Psi/dt$ , then show that the ONLY change in the governing equations — (1.8), (1.9), and (1.11) — is that  $\Omega$  is no longer a constant, and is replaced by  $\Omega = \Omega(t)$ , as defined above.

**Example 1.3** Consider equation (1.9) for the case of a variable, and periodic, rotation rate  $\Omega = \Omega(t)$ . Assume that the amplitude of  $\Omega$  remains below  $\omega$  for all times:  $0 \leq \Omega^2 < \omega^2$ . In this case, intuition would seem to indicate that  $\Theta \equiv 0$  is a stable solution to the equation, with small perturbations near it obeying the linear equation:

$$\frac{d^2\Theta}{dt^2} + (\omega^2 - \Omega^2) \Theta = 0. \quad (1.12)$$

However, it turns out that, when the period of the forcing  $\Omega$  is appropriately selected, it is possible to make the solutions of (1.12) grow in time — thus turning the solution  $\Theta \equiv 0$  unstable.<sup>1</sup> This phenomena is known as a **parametric instability**, and we will study it later — when we cover **Floquet theory**. When nonlinearity and dissipation are added to the system, chaotic behavior can result from it.

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<sup>1</sup>This can happen even if the amplitude of  $\Omega$  is small.

**Remark 1.3 — Derivation of the Euler-Lagrange equation (1.5).**

Let  $\delta\mathcal{S} = \delta\mathcal{S}(t)$  be a small (but arbitrary) perturbation to the mechanical trajectory  $\mathcal{S}$ , vanishing at  $t_1$  and  $t_2$ . Let  $\delta\mathcal{L}$  be the corresponding perturbation to the action:

$$\begin{aligned}\delta\mathcal{L} &= \mathcal{L}[\mathcal{S} + \delta\mathcal{S}] - \mathcal{L}[\mathcal{S}] = \int_{t_1}^{t_2} \left( L(\dot{\mathcal{S}} + \delta\dot{\mathcal{S}}, \mathcal{S} + \delta\mathcal{S}, t) - L(\dot{\mathcal{S}}, \mathcal{S}, t) \right) dt \\ &= \int_{t_1}^{t_2} \left( L_2(\dot{\mathcal{S}}, \mathcal{S}, t) \delta\mathcal{S} + L_1(\dot{\mathcal{S}}, \mathcal{S}, t) \delta\dot{\mathcal{S}} \right) dt + O(\delta\mathcal{S}^2) \\ &= \int_{t_1}^{t_2} \left( L_2(\dot{\mathcal{S}}, \mathcal{S}, t) - \frac{d}{dt} \left( L_1(\dot{\mathcal{S}}, \mathcal{S}, t) \right) \right) \delta\mathcal{S} dt + O(\delta\mathcal{S}^2),\end{aligned}$$

where  $L_1 = \frac{\partial L}{\partial \dot{\mathcal{S}}}$ ,  $L_2 = \frac{\partial L}{\partial \mathcal{S}}$ , and we have integrated by parts in the third line. Because  $\mathcal{S}$  is a stationary point, the  $O(\delta\mathcal{S})$  terms in  $\delta\mathcal{L}$  must vanish for **all** possible choices of  $\delta\mathcal{S}$ . It follows that

$$L_2(\dot{\mathcal{S}}, \mathcal{S}, t) - \frac{d}{dt} \left( L_1(\dot{\mathcal{S}}, \mathcal{S}, t) \right) = 0,$$

which is exactly the Euler-Lagrange equation (1.5).

**Problem 1.3** Consider the case of a hoop that is restricted to move up and down on a vertical plane. Derive the equation for the bead motion in this case. Parameterize the wire in the form

$$X = R \sin\left(\frac{s}{R}\right), \quad Y \equiv 0, \quad \text{and} \quad Z = -R \cos\left(\frac{s}{R}\right) + A \sin(\Omega t), \quad (1.13)$$

where  $R$  is the radius of the hoop (constant),  $\Omega$  is the (constant) angular frequency of the up-down motions of the hoop, and  $A$  is the (constant) amplitude of the hoop oscillations. Then show that the equation for the bead position, written in terms of the angle  $\Theta = \mathcal{S}/R$  that the bead makes with the bottom point on the hoop, is given by:

$$\frac{d^2\Theta}{dt^2} + \left( \omega^2 - \epsilon\Omega^2 \sin(\Omega t) \right) \sin \Theta = 0, \quad (1.14)$$

where  $\omega^2 = g/R$ , and  $\epsilon = A/R$ . For small values of  $\Theta$ , the linearization of this equation is of the same form as equation (1.12). Thus this is another example where parametric instability can occur. Furthermore, consider the linearization of this equation near the equilibrium solution  $\Theta \equiv \pi$  — bead on the top end of the hoop. Then, for  $\Theta = \pi + \psi$  and  $\psi$  small, we can write:

$$\frac{d^2\psi}{dt^2} - \left( \omega^2 - A\Omega^2 \sin(\Omega t) \right) \psi = 0. \quad (1.15)$$

Intuitively we expect the  $\Theta \equiv \pi$  solution to be unstable. However, it is possible to select the frequency and amplitude of the up-down oscillations so that this solution becomes stable. This is an example of **parametric stabilization** — to be covered later in the course.

**Example 1.4** Consider now<sup>2</sup> a wire restricted to move on a fixed vertical plane — so that  $Y \equiv 0$  in equation (1.1). Furthermore, assume that the wire position is given in the form

$$z = F(x, t). \quad (1.16)$$

What is then the equation of motion, with the bead position given as  $x = \chi(t)$ ? To find this equation, we could start from equation (1.6), and transform variables. This, however, is quite messy. It is much easier to re-write the Lagrangian  $L$  in (1.4) in terms of the new variables, and then use again the Euler-Lagrange equation. Clearly, we have:

$$L = \frac{1}{2} M (1 + F_x^2) \dot{\chi}^2 + M F_x F_t \dot{\chi} - \left( g M F - \frac{1}{2} M F_t^2 \right) = \frac{1}{2} \mu \dot{\chi}^2 + \kappa \dot{\chi} - V, \quad (1.17)$$

where  $F$  and its derivatives are evaluated at  $(\chi, t)$ . Furthermore:  $\mu = \mu(\chi, t) = M(1 + F_x^2)$ ,

$\kappa = \kappa(\chi, t) = M F_x F_t$ , and  $V = V(\chi, t) = g M F - (1/2) M F_t^2$ . The Euler-Lagrange equation

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\chi}} \right) - \frac{\partial L}{\partial \chi} = 0$$

then yields:

$$\frac{d}{dt} \left( \mu \frac{d\chi}{dt} \right) = \frac{1}{2} \mu_{\chi} \left( \frac{d\chi}{dt} \right)^2 - V_{\chi} - \kappa_t. \quad (1.18)$$

## 2 Add friction to the equations.

We want now to add the effects of friction to the equation of motion for the bead. The friction laws for solid-to-solid contact are quite complicated, thus we will simplify the problem by assuming that the wire is covered by a thin layer of lubricant liquid. In this case, an adequate model is provided by: friction produces a force opposing the motion, of magnitude proportional to the velocity of the bead relative to the wire. Since  $s$  is not just arclength, but a material

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<sup>2</sup>The example in problem 1.3 is of this type.

coordinate along the wire, the sliding velocity of the bead along the wire is simply  $\dot{\mathcal{S}}$ . Thus we modify equation (1.6) to:

$$M \frac{d^2 \mathcal{S}}{dt^2} = -\nu \frac{d\mathcal{S}}{dt} - \frac{\partial V}{\partial \mathcal{S}}(\mathcal{S}, t) - \frac{\partial \kappa}{\partial t}(\mathcal{S}, t), \quad (2.1)$$

where  $\nu$  is the friction coefficient — which we assume is a constant. This is the **equation for a bead moving along a lubricated rigid moving wire, under the influence of gravity, with the friction force included.**

**Example 2.1** Add friction to the rotating hoop in example 1.1. *It is easy to see that equation (1.9) is modified to:*

$$\frac{d^2 \Theta}{dt^2} + \frac{\nu}{M} \frac{d\Theta}{dt} + (\omega^2 - \Omega^2 \cos \Theta) \sin \Theta, \quad (2.2)$$

by the addition of friction. When the rotation rate is variable,  $\Omega = \Omega(t)$  — see problem 1.2. More generally, equation (1.11) for a rotating wire is modified to:

$$\frac{d^2 \mathcal{S}}{dt^2} + \frac{\nu}{M} \frac{d\mathcal{S}}{dt} + gZ'(\mathcal{S}) - \Omega^2 R(\mathcal{S})R'(\mathcal{S}) = 0. \quad (2.3)$$

**Example 2.2** Consider the situation in example 1.4 above. *To add the effects of friction when the equation of motion is written as in (1.18), we need to compute first what the velocity of the bead is relative to the wire. This is not as simple as it may appear, for equation (1.16) does not provide enough information as to the actual motion of the wire mass points — for example: the wire may have a fixed shape, with the points in the wire “sliding” along this shape.<sup>3</sup> Thus extra information is needed. For example, let us assume that there is some material point in the wire for which  $x = x_0$  is constant as the wire deforms and moves. In this case we can proceed as follows:*

$$\begin{aligned} ds^2 &= dx^2 + dz^2 = (1 + F_x^2) dx^2 = \alpha^2 dx^2, & \implies \\ \mathcal{S} &= \int_{x_0}^x \alpha(x, t) dx, & \implies \\ \frac{d\mathcal{S}}{dt} &= \alpha(\chi, t) \frac{d\chi}{dt} + \int_{x_0}^x \alpha_t(x, t) dx, & \implies \\ \mathcal{F} &= \nu \left( \alpha(\chi, t) \frac{d\chi}{dt} + \int_{x_0}^x \alpha_t(x, t) dx \right), & (2.4) \end{aligned}$$

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<sup>3</sup>Think of a circular wire rotating around its center. See example 2.3.



where  $\alpha = \alpha(x, t) = \sqrt{1 + F_x^2}$  (thus  $\mu = M\alpha^2$ , and  $\alpha\alpha_t = F_x F_t$ ), and  $\mathcal{F}$  is the magnitude of the friction force. The magnitude of the component of the friction force along the  $x$ -axis is thus given by  $\mathcal{F}/\alpha$ . This component must oppose the inertial force along the  $x$ -axis, which is given by  $M\ddot{\chi}$ . Since  $\mu = M\alpha^2$ , we conclude that equation (1.18) must be modified as follows:

$$\frac{d}{dt} \left( \mu \frac{d\chi}{dt} \right) + \alpha\mathcal{F} = \frac{1}{2} \mu_{\chi} \left( \frac{d\chi}{dt} \right)^2 - V_{\chi} - \kappa_t, \quad (2.5)$$

where  $\mathcal{F} = \mathcal{F}(\dot{\chi}, \chi, t)$  is given by equation (2.4). A particularly simple instance of this occurs in the case of a wire moving rigidly up and down. In this case we can write, for  $F$  in (1.16),

$$z = F(x, t) = f(x) + h(t). \quad (2.6)$$

Then  $\mu = \mu(\chi) = M \left( 1 + (f'(\chi))^2 \right)$ ,  $\kappa = \kappa(\chi, t) = Mf'(\chi)\dot{h}(t)$ ,  $\alpha = \alpha(\chi) = \sqrt{1 + (f'(\chi))^2}$ ,  $V = V(\chi, t) = gMf(\chi) - \left( gMh(t) + \frac{1}{2}M(\dot{h}(t))^2 \right)$ , and  $\mathcal{F} = \nu\alpha\dot{\chi}$ . Thus, the equation of motion can be written in the form:

$$\frac{d}{dt} \left( \alpha \frac{d\chi}{dt} \right) + \frac{\nu}{M} \alpha \frac{d\chi}{dt} + \frac{1}{\alpha} f'(\chi) (g + \ddot{h}(t)) = 0. \quad (2.7)$$

**Example 2.3 Sliding wires and friction.** We consider two examples. First a horizontal straight wire sliding at a variable speed. Thus:

$$X = s + u(t), \quad \text{and} \quad Y = Z = 0. \quad (2.8)$$

Let  $v = \frac{du}{dt}$  be the wire speed, and  $a = \frac{dv}{dt}$  be the wire acceleration. Then the formulas above equation (1.6) yield:  $\kappa = Mv$ ,  $\beta = \frac{1}{2}Mv^2$ , and  $V = -\frac{1}{2}Mv^2$ . Thus, from equation (2.1)

$$\frac{d^2\mathcal{S}}{dt^2} + \frac{\nu}{M} \frac{d\mathcal{S}}{dt} + a = 0 \quad \iff \quad \frac{d^2\chi}{dt^2} + \frac{\nu}{M} \left( \frac{d\chi}{dt} - v \right) = 0, \quad (2.9)$$

where  $\chi = \mathcal{S} + u$ . The second example is that of a horizontal circular wire rotating around its center. Thus

$$X = R \cos \theta, \quad Y = R \sin \theta, \quad \text{and} \quad Z = 0, \quad (2.10)$$

where  $R$  is the radius of the hoop,  $\theta = s/R + \phi$ , and  $\phi = \phi(t)$  indicates the wire motion. Let  $\Omega = \frac{d\phi}{dt}$  be the wire angular velocity. Then equation (2.1) yields, for  $\Theta = \frac{\mathcal{S}}{R} + \phi$ ,

$$\frac{d^2\Theta}{dt^2} + \frac{\nu}{M} \left( \frac{d\Theta}{dt} - \Omega \right) = 0. \quad (2.11)$$

### 3 Friction dominates inertia.

#### 3.1 Nondimensional equations.

We consider equation (2.1), for a simple situation where the equation is autonomous (no time dependence). Thus, the equation takes the form

$$M \frac{d^2 \mathcal{S}}{dt^2} + \nu \frac{d\mathcal{S}}{dt} = F(\mathcal{S}), \quad (3.1)$$

where  $F$  is the force driving the bead along the wire (produced by gravity and the wire motion). Assume now that  $F$  varies on some characteristic length scale, and has a typical size. Thus we can write

$$F = F_0 \Psi \left( \frac{\mathcal{S}}{L} \right) \quad (3.2)$$

where  $F_0$  is a constant with force dimensions,  $L$  is a constant with dimensions of length, and  $\Psi$  is nondimensional and has  $O(1)$  size, with an  $O(1)$  derivative. We introduce nondimensional variables as follows:

$$\Theta = \frac{\mathcal{S}}{L} \quad \text{and} \quad \tau = \frac{F_0}{\nu L} t, \quad (3.3)$$

where  $\frac{\nu L}{F_0}$  is the time scale that results from the balance of the applied forces and the viscous forces. In terms of these variables, the equation becomes:

$$\epsilon \frac{d^2 \Theta}{d\tau^2} + \frac{d\Theta}{d\tau} = \Psi(\Theta), \quad (3.4)$$

where  $\epsilon = \frac{MF_0}{L\nu^2}$  is a nondimensional parameter.

#### 3.2 Limit of large friction.

If the inertial forces are negligible relative to the viscous forces, to be precise, if  $0 < \epsilon \ll 1$  in (3.4), the inertial terms in the equation can be neglected. This leads to:

$$\frac{d\Theta}{d\tau} = \Psi(\Theta). \quad (3.5)$$

We point out that it is, generally, dangerous to neglect terms that involve the highest derivative in an equation — even if the term is multiplied by a very small parameter.

In the particular case here, this can be entirely justified,<sup>4</sup> but (in general) one must be very careful.

## 4 Problem Answers.

The problem answers will be handed out separately, with the solutions to problem sets and exams.

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<sup>4</sup>This will be done later in the course.